# Price manipulation in models of the order book

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# Background

- In previous work, we showed how the modeling of market impact is constrained by requiring no-price-manipulation.
- In particular:
  - If the decay of market impact is exponential, market impact must be linear in quantity.
  - If decay of market impact is power-law and sensitivity to quantity is also power-law, no-price-manipulation imposes inequality constraints on the exponents.

## Price process

- We suppose that the stock price $S_t$ at time $t$ is given by

$$S_t = S_0 + \int_0^t f(\dot{x}_s)\, G(t-s)\, ds + \int_0^t \sigma\, dZ_s \tag{1}$$

  where $\dot{x}_s$ is our rate of trading in dollars at time $s < t$, $f(\dot{x}_s)$ represents the impact of trading at time $s$ and $G(t-s)$ is a decay factor.
- $S_t$ follows an arithmetic random walk with a drift that depends on the accumulated impacts of previous trades.
- The cumulative impact of (others') trading is implicitly in $S_0$ and the noise term.
- Drift is ignored.

- We refer to $f(\cdot)$ as the *instantaneous market impact function* and to $G(\cdot)$ as the decay kernel.
- (1) is a generalization of processes previously considered by Almgren, Bouchaud and Obizhaeva and Wang.

**Remark**

The price process (1) is not the only possible generalization of price processes considered previously. On the one hand, it seems like a natural generalization. On the other hand, it is not motivated by any underlying model of the order book.

# Model as limit of discrete time process

- The continuous time process (1) can be viewed as a limit of a discrete time process (see Bouchaud et al. for example):

$$S_t = \sum_{i<t} f(\delta x_i)\, G(t-i) + \text{ noise}$$

where $\delta x_i = \dot{x}_i \, \delta t$ is the quantity traded in some small time interval $\delta t$ characteristic of the stock, and by abuse of notation, $f(\cdot)$ is the market impact function.

  - $\delta x_i > 0$ represents a purchase and $\delta x_i < 0$ represents a sale.
  - $\delta t$ could be thought of as $1/\nu$ where $\nu$ is the trade frequency.
  - Increasing the rate of trading $\dot{x}_i$ is equivalent to increasing the quantity traded each $\delta t$.

# Cost of trading

- Denote the number of shares outstanding at time $t$ by $x_t$. Then from (1), the cost $C[\Pi]$ associated with a given trading strategy $\Pi = \{x_t\}$ is given by

$$C[\Pi] = \int_0^T \dot{x}_t \, dt \int_0^t f\left(\dot{x}_s\right) \, G(t-s) \, ds \tag{2}$$

- The $dx_t = \dot{x}_t \, dt$ shares liquidated at time $t$ are traded on average at a price

$$S_t = S_0 + \int_0^t f\left(\dot{x}_s\right) \, G(t-s) \, ds$$

which reflects the residual cumulative impact of all prior trading.

# The principle of No Price Manipulation

A trading strategy $\Pi = \{x_t\}$ is a *round-trip trade* if

$$\int_0^T \dot{x}_t \, dt = 0$$

We define a *price manipulation* to be a round-trip trade $\Pi$ whose expected cost $C[\Pi]$ is negative.

**The principle of no-price-manipulation**

Price manipulation is not possible.

**Remark**

If price manipulation were possible, the optimal strategy would not exist.

# A specific strategy

Consider a strategy where shares are accumulated at the (positive) constant rate $v_1$ and then liquidated again at the (positive) constant rate $v_2$. According to equation (2), the cost of this strategy is given by $C_{11} + C_{22} - C_{12}$ with

$$\begin{aligned} C_{11} &= v_1 \, f(v_1) \int_0^{\theta T} dt \int_0^t G(t-s)\, ds \\ C_{22} &= v_2 \, f(v_2) \int_{\theta T}^{T} dt \int_{\theta T}^{t} G(t-s)\, ds \\ C_{12} &= v_2 \, f(v_1) \int_{\theta T}^{T} dt \int_0^{\theta T} G(t-s)\, ds \end{aligned} \tag{3}$$

where $\theta$ is such that $v_1 \, \theta \, T - v_2 \, (T - \theta \, T) = 0$ so

$$\theta = \frac{v_2}{v_1 + v_2}$$

# Special case: Trade in and out at the same rate

One might ask what happens if we trade into, then out of a position at the same rate $v$. If $G(\cdot)$ is strictly decreasing,

$$
\begin{aligned}
C[\Pi] &= v\,f(v)\left\{\int_0^{T/2} dt \int_0^t G(t-s)\,ds + \int_{T/2}^{T} dt \int_{T/2}^{t} G(t-s)\,ds \right. \\
&\quad \left. - \int_{T/2}^{T} dt \int_0^{T/2} G(t-s)\,ds \right\} \\
&= v\,f(v)\left\{\int_0^{T/2} dt \int_0^t [G(t-s) - G(t+T/2-s)]\,ds \right. \\
&\quad \left. + \int_{T/2}^{T} dt \int_{T/2}^{t} [G(t-s) - G(T-s)]\,ds \right\} > 0
\end{aligned}
$$

- We conclude that if price manipulation is possible, it must involve trading in and out at different rates.

# Exponential decay

Suppose that the decay kernel has the form

$$G(\tau) = e^{-\rho\,\tau}$$

Then, explicit computation of all the integrals in (3) gives

$$\begin{aligned}
C_{11} &= v_1\, f\,(v_1)\,\frac{1}{\rho^2}\,\left\{e^{-\rho\,\theta\,T} - 1 + \rho\,\theta\,T\right\} \\
C_{12} &= v_2\, f\,(v_1)\,\frac{1}{\rho^2}\,\left\{1 + e^{-\rho\,T} - e^{-\rho\,\theta\,T} - e^{-\rho\,(1-\theta)\,T}\right\} \\
C_{22} &= v_2\, f\,(v_2)\,\frac{1}{\rho^2}\,\left\{e^{-\rho\,(1-\theta)\,T} - 1 + \rho\,(1-\theta)\,T\right\} \qquad (4)
\end{aligned}$$

We see in particular that the no-price-manipulation principle forces a relationship between the instantaneous impact function $f(\cdot)$ and the decay kernel $G(\cdot)$.

# Exponential decay

After making the substitution $\theta = v_2/(v_1 + v_2)$ and imposing the principle of no-price-manipulation, we obtain

$$\begin{aligned} & v_1\, f(v_1)\, \left[e^{-\frac{v_2\,\rho}{v_1+v_2}} - 1 + \frac{v_2\,\rho}{v_1 + v_2}\right] \\ & + v_2\, f(v_2)\, \left[e^{-\frac{v_1\,\rho}{v_1+v_2}} - 1 + \frac{v_1\,\rho}{v_1 + v_2}\right] \\ & - v_2\, f(v_1)\, \left[1 + e^{-\rho} - e^{-\frac{v_1\,\rho}{v_1+v_2}} - e^{-\frac{v_2\,\rho}{v_1+v_2}}\right] \geq 0 \end{aligned} \tag{5}$$

where, without loss of generality, we have set $T = 1$. We note that the first two terms are always positive so price manipulation is only possible if the third term $(C_{12})$ dominates the others.

# Example: $f(v) = \sqrt{v}$

Let $v_1 = 0.2$, $v_2 = 1$, $\rho = 1$. Then the cost of liquidation is given by

$$C = C_{11} + C_{22} - C_{12} = -0.001705 < 0$$

Since $\rho$ really represents the product $\rho\, T$, we see that for any choice of $\rho$, we can find a combination $\{v_1, v_2, T\}$ such that a round trip with no net purchase or sale of stock is profitable. We conclude that if market impact decays exponentially, the no-price-manipulation principle excludes a square root instantaneous impact function.

Can we generalize this?

## Expansion in $\rho$

Expanding expression (5) in powers of $\rho$, we obtain

$$\frac{v_1\, v_2\, \left[v_1\, f(v_2) - v_2\, f(v_1)\right]\, \rho^2}{2(v_1+v_2)^2} + O\left(\rho^3\right) \geq 0$$

We see that price manipulation is always possible for small $\rho$ unless $f(v)$ is linear in $v$ and we have

**Lemma**

*Under exponential decay of market impact, no-price-manipulation implies $f(v) \propto v$.*

# Empirical viability of exponential decay of market impact

- Empirically, market impact is concave in $v$ for small $v$.
- Also, market impact must be convex for very large $v$
  - Imagine submitting a sell order for 1 million shares when there are bids for only 100,000.
- We conclude that the principle of no-price-manipulation excludes exponential decay of market impact for any empirically reasonable instantaneous market impact function $f(\cdot)$.

# Power-law decay

Suppose now that the decay kernel has the form

$$G(t-s)=\frac{1}{(t-s)^{\gamma}},\ 0<\gamma<1$$

Then, explicit computation of all the integrals in (3) gives

$$\begin{aligned}
C_{11} &= v_1\, f\,(v_1)\,\frac{T^{2-\gamma}}{(1-\gamma)\,(2-\gamma)}\,\theta^{2-\gamma} \\
C_{22} &= v_2\, f\,(v_2)\,\frac{T^{2-\gamma}}{(1-\gamma)\,(2-\gamma)}\,(1-\theta)^{2-\gamma} \\
C_{12} &= v_2\, f\,(v_1)\,\frac{T^{2-\gamma}}{(1-\gamma)\,(2-\gamma)}\,\left\{1-\theta^{2-\gamma}-(1-\theta)^{2-\gamma}\right\}
\end{aligned} \tag{6}$$

# Power-law decay

According to the principle of no-price-manipulation , substituting $\theta = v_2/(v_1 + v_2)$, we must have

$$f(v_1)\left\{v_1\, {v_2}^{1-\gamma} - (v_1+v_2)^{2-\gamma} + {v_1}^{2-\gamma} + {v_2}^{2-\gamma}\right\} + f(v_2)\, {v_1}^{2-\gamma} \geq 0 \tag{7}$$

- If $\gamma = 0$, the no-price-manipulation condition (7) reduces to

  $$f(v_2)\, v_1 - f(v_1)\, v_2 \geq 0$$

  so again, permanent impact must be linear.
- If $\gamma = 1$, equation (7) reduces to

  $$f(v_1) + f(v_2) \geq 0$$

  So long as $f(\cdot) \geq 0$, there is no constraint on $f(\cdot)$ when $\gamma = 1$.

# The limit $v_1 \ll v_2$ and $0 < \gamma < 1$

In this limit, we accumulate stock much more slowly than we liquidate it. Let $v_1 = \epsilon\, v$ and $v_2 = v$ with $\epsilon \ll 1$. Then, in the limit $\epsilon \to 0$, with $0 < \gamma < 1$, equation (7) becomes

$$\begin{aligned} & f(\epsilon\, v)\,\left\{\epsilon - (1+\epsilon)^{2-\gamma} + \epsilon^{2-\gamma} + 1\right\} + f(v)\,\epsilon^{2-\gamma} \\ \sim\quad & -f(\epsilon\, v)\,(1-\gamma)\,\epsilon + f(v)\,\epsilon^{2-\gamma} \geq 0 \end{aligned}$$

so for $\epsilon$ sufficiently small we have

$$\frac{f\,(\epsilon\, v)}{f\,(v)} \leq \frac{\epsilon^{1-\gamma}}{1-\gamma} \tag{8}$$

If the condition (8) is not satisfied, price manipulation is possible by accumulating stock slowly, maximally splitting the trade, then liquidating it rapidly.

# Power-law impact: $f(v) \propto v^\delta$

If $f(v) \sim v^\delta$ (as per Almgren et al.), the no-price-manipulation condition (8) reduces to

$$\epsilon^{1-\gamma-\delta} \geq 1 - \gamma$$

and we obtain

**Small $v$ no-price-manipulation condition**

$$\gamma + \delta \geq 1$$

# Cost of VWAP with power-law market impact and decay

From equation (6), the cost of an interval VWAP execution with duration $T$ is proportional to

$$C = v\, f\,(v)\;\, T^{2-\gamma}$$

Noting that $v = n/T$, and putting $f(v) \propto (v/V)^{\delta}$, the cost per share is proportional to

$$\left(\frac{n}{V}\right)^{\delta}\; T^{1-\gamma-\delta}$$

If $\gamma + \delta = 1$, the cost per share is *independent* of $T$ and in particular, if $\gamma = \delta = 1/2$, the impact cost per share is proportional to $\sqrt{n/V}$, which is the well-known square-root formula for market impact as described by, for example, Grinold and Kahn.

# The square-root formula, $\gamma$ and $\delta$

- The square-root market impact formula has been widely used in practice for many years.
- If correct, this formula implies that the cost of liquidating a stock is independent of the time taken.
    - Fixing market volume and volatility, impact depends only size.
- We can check this prediction empirically.
    - See for example Engle, Ferstenberg and Russell, 2008.
- Also, according to Almgren, $\delta \approx 0.6$ and according to Bouchaud $\gamma \approx 0.4$.

**Empirical observation**

$$\delta + \gamma \approx 1!$$

## The shape of the order book

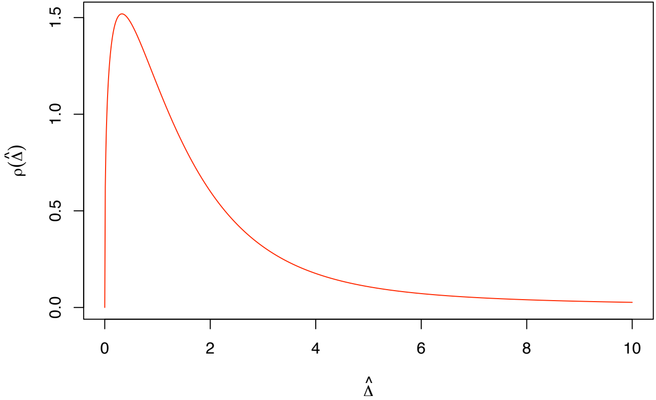
Bouchaud, Mézard and Potters (2002) derive the following approximation to the average density $\rho(\hat{\Delta})$ of orders as a function of a rescaled distance $\hat{\Delta}$ from the price level at which the order is placed to the current price:

$$\rho(\hat{\Delta}) = e^{-\hat{\Delta}} \int_0^{\hat{\Delta}} du \, \frac{\sinh(u)}{u^{1+\mu}} + \sinh(\hat{\Delta}) \int_{\hat{\Delta}}^{\infty} du \, \frac{e^{-u}}{u^{1+\mu}} \tag{9}$$

where $\mu$ is the exponent in the empirical power-law distribution of new limit orders.

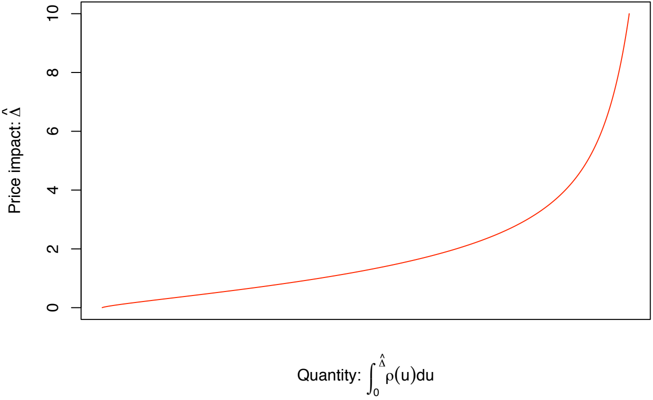
# Approximate order density

The red line is a plot of the order density $\rho(\hat{\Delta})$ with $\mu = 0.6$ (as estimated by Bouchaud, Mézard and Potters).



# Virtual price impact

Switching $x-$ and $y-$axes in a plot of the cumulative order density gives the virtual impact function plotted below. The red line corresponds to $\mu = 0.6$ as before.



# Impact for high trading rates

- You can't trade more than the total depth of the book so price impact increases without limit as $n \to n_{max}$.
- For a sufficiently large trading rate $v$, it can be shown that

$$f(v) \sim \frac{1}{(1 - v/v_{max})^{1/\mu}}$$

- Setting $v = v_{max}(1 - \epsilon)$ and taking the limit $\epsilon \to 0$,

$$f(v) \sim \frac{1}{\epsilon^{1/\mu}} \text{ as } \epsilon \to 0.$$

- Imagine we accumulate stock at a rate close to $v_{max} := 1$ and liquidate at some (lower) rate $v$.
  - This is the pump and dump strategy!

# Impact for high trading rates continued

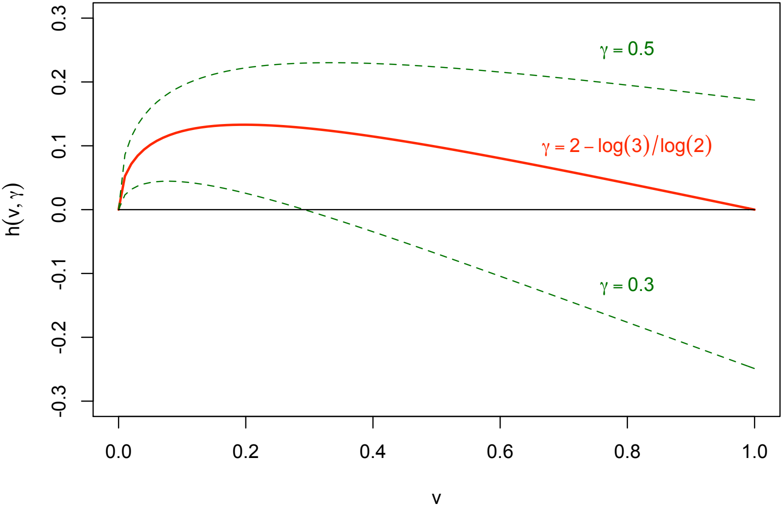
- Substituting into condition (7) gives

$$\frac{1}{\epsilon^{1/\mu}}\left\{(1-\epsilon)\,v^{1-\gamma}-(1-\epsilon+v)^{2-\gamma}+(1-\epsilon)^{2-\gamma}+v^{2-\gamma}\right\}$$
$$+f\,(v)\,(1-\epsilon)^{2-\gamma}\geq 0$$

- No-price-manipulation imposes that

$$h(v,\,\gamma):=v^{1-\gamma}-(1+v)^{2-\gamma}+1+v^{2-\gamma}\geq 0$$

# Graphical illustration



# Large size no price manipulation condition

From the picture, we see that $h(v, \gamma) \geq 0$ implies

**Large size no price manipulation condition**

$$\gamma > \gamma^* = 2 - \frac{\log 3}{\log 2}$$

# Long memory of order flow

- It is empirically well-established that order-flow is a long memory process.
  - More precisely, the autocorrelation function of order signs decays as a power-law.
  - There is evidence that this autocorrelation results from order splitting.
- Imposing linear growth of return variance in trading time (Bouchaud et al. 2004) in an effective model such as (1) forces power-law decay of market impact with exponent $\gamma \leq 1/2$.

# Summary of prior work

- By imposing the principle of no-price-manipulation, we showed that if market impact decays as a power-law $1/(t-s)^\gamma$ and the instantaneous market impact function is of the form $f(v) \propto v^\delta$, we must have

$$\gamma + \delta \geq 1$$

- We excluded the combination of exponential decay with nonlinear market impact.

- We observed that if the average cost of a (not-too-large) VWAP execution is roughly independent of duration, the exponent $\delta$ of the power law of market impact should satisfy:

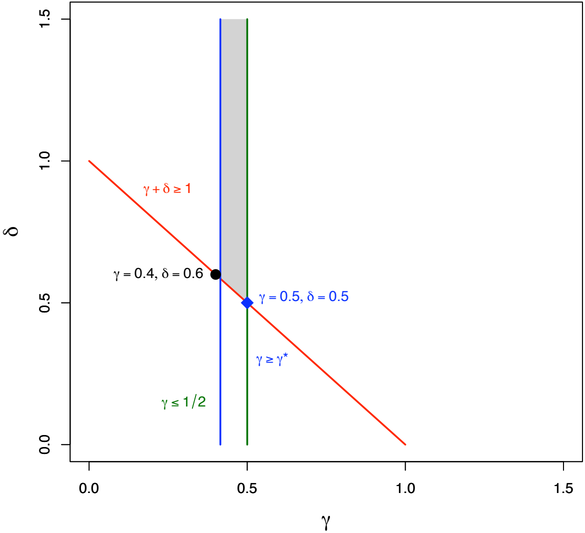
$$\delta + \gamma \approx 1$$

- By considering the tails of the limit-order book, we deduce that

$$\gamma \geq \gamma^* := 2 - \frac{\log 3}{\log 2} \approx 0.415$$

- Long memory of order flow imposes $\gamma \leq 1/2$.

# Schematic presentation of constraints



# The model of Alfonsi, Fruth and Schied

Alfonsi, Fruth and Schied (2009) consider the following (AS) model of the order book:

- There is a continuous (in general nonlinear) density of orders $f(x)$ above some martingale ask price $A_t$. The cumulative density of orders up to price level $x$ is given by

$$F(x) := \int_0^x f(y)\,dy$$

- Executions eat into the order book (*i.e.* executions are with market orders).
- A purchase of $\xi$ shares at time $t$ causes the ask price to increase from $A_t + D_t$ to $A_t + D_{t+}$ with

$$\xi = \int_{D_t}^{D_{t+}} f(x)\,dx = F(D_{t+}) - F(D_t)$$

- We can define a volume impact process

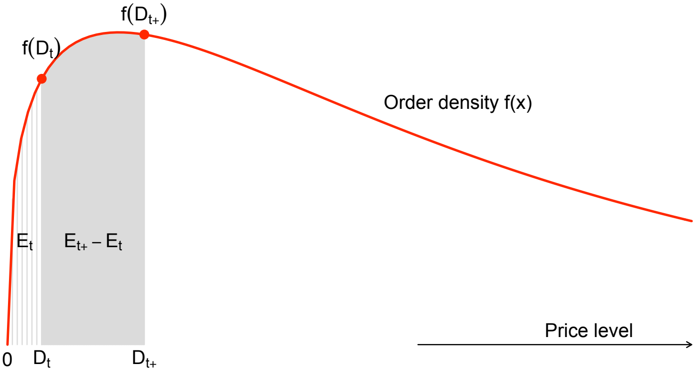
$$E_t := F(D_t)$$

which represents how much of the book has been eaten up by executions up to time $t$.

- Depending on the model, either the spread $D_t$ or the volume impact process $E_t$ revert exponentially at some rate $\rho$.

  This captures the resiliency of the order book: Limit orders arrive to replenish order density lost through executions.

# Schematic of the model

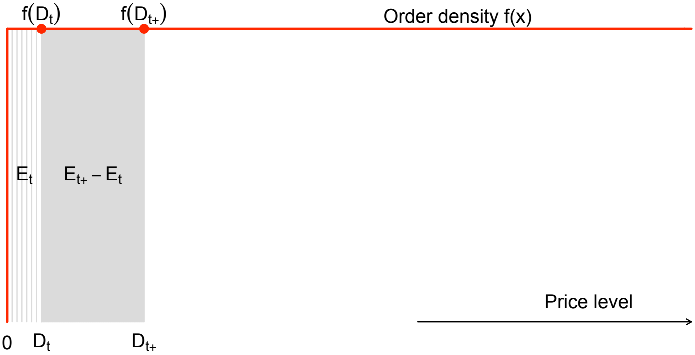



When a trade of size $\xi$ is placed at time $t$,

$$\begin{aligned} E_t &\mapsto E_{t+} = E_t + \xi \\ D_t = F^{-1}(E_t) &\mapsto D_{t+} = F^{-1}(E_{t+}) = F^{-1}(E_t + \xi) \end{aligned}$$

# Example: The model of Obizhaeva and Wang

Obizhaeva and Wang (2005) consider a block-shaped order book with constant order density.



In the Obizhaeva Wang (OW) model, we thus have

$$
\begin{aligned}
f(x) &= 1 \\
F(x) &= x \\
E_t &= D_t \\
\Delta D_t &:= D_{t+} - D_t = \xi
\end{aligned}
$$

Thus market impact $\Delta D_t$ is linear in the quantity $\xi$. Between executions, the spread $D_t$ and the volume impact process $E_t$ both decay exponentially at rate $\rho$ so that at time $t$ after an execution at the earlier time $s$, we have

$$D_t = D_{s+}\, e^{-\rho\,(t-s)}$$

## Cost of execution and optimal trading strategy

Given a trading strategy $\Pi$ with trading at the rate $\dot{x}_t$, the cost of execution in the OW model is given by

$$\mathcal{C}(\Pi) = \int_0^T \dot{x}_t \, dt \int_0^t \dot{x}_s \, e^{-\rho\,(t-s)} \, ds$$

When the trading policy $\dot{x}_t = u_t$ is statically optimal, the Euler-Lagrange equation applies:

$$\frac{\partial}{\partial t} \frac{\delta \mathcal{C}}{\delta u_t} = 0$$

Then, for some constant $A$, we have

$$\frac{\delta \mathcal{C}}{\delta u_t} = \int_0^t u_s \, e^{-\rho\,(t-s)} \, ds + \int_t^T u_s \, e^{-\rho\,(s-t)} \, ds = A \qquad (10)$$

Substituting

$$u_t = \delta(t) + \rho + \delta(T - t)$$

where $\delta(\cdot)$ is the Dirac delta function into (10) gives

$$\begin{aligned}
\frac{\delta \mathcal{C}}{\delta u_t} &= \int_0^t u_s \, e^{-\rho\,(t-s)} \, ds + \int_t^T u_s \, e^{-\rho\,(s-t)} \, ds \\
&= e^{-\rho\,t} + \int_0^t \rho \, e^{-\rho\,(t-s)} \, ds + \int_t^T \rho \, e^{-\rho\,(s-t)} \, ds + e^{-\rho\,(T-t)} \\
&= 2
\end{aligned}$$

so $u_t$ is the optimal strategy.

**The optimal strategy**

Trade blocks of stock at times $t = 0$ and $t = T$ and trade continuously at rate $\rho$ between these two times.

# No price manipulation

- The optimal strategy involves only purchases of stock, no sales.
- Thus there cannot be price manipulation in the OW model.
- The OW price process is a special case of (1) with linear price impact and exponential decay of market impact.
  - Consistent with our lemma, there is no price manipulation.
- The OW model is also a special case of the AS model.

# Cost of execution in the AS model

In general, the cost of trade execution in the AS model has a continuous piece and a discrete piece:

$$\mathcal{C}(\Pi) = \int_0^T \dot{x}_t \, F^{-1}(E_t) \, dt + \sum_{t \leq T} \left[ H(E_{t+}) - H(E_t) \right] \tag{11}$$

where

$$H(x) = \int_0^x F^{-1}(x) \, dx$$

and

$$E_t = \int_0^t \dot{x}_s \, e^{-\rho\,(t-s)} \, ds$$

is the volume impact process.

# Optimal strategy in the AS model

**The optimal strategy**

Trade blocks of stock at times $t = 0$ and $t = T$ and trade continuously between these two times.

The optimal size $\xi_0$ of the initial block purchase satisfies

$$F^{-1}(X - \xi_0\,\rho\,T) = F^{-1}(\xi_0) + F^{-1\prime}(\xi_0)\,\xi_0$$

The optimal continuous trading rate is $\rho\,\xi_0$ and the optimal size of the final block is just

$$\xi_T = X - \left(\xi_0 + \int_{0^+}^{T^-} \rho\,\xi_0\,dt\right) = X - \xi_0\,(1 + \rho\,T)$$

# No price manipulation

- Once again, the optimal strategy involves only purchases of stock, no sales.
- Thus there cannot be price manipulation in the AS model.

**What's going on?**

"Under exponential decay of market impact, no-price-manipulation implies $f(v) \propto v$"

# A potential conundrum

## From Alfonsi and Schied (2009)

"Our result on the non-existence of profitable price manipulation strategies strongly contrasts Gatheral's conclusion that the widely-assumed exponential decay of market impact is compatible only with linear market impact."

"The preceding corollary shows that, in our Model 1, exponential resilience of price impact is well compatible with nonlinear impact ...

This fact is in stark contrast to Gatheral's observation that, in a related but different continuous-time model, exponential decay of price impact gives rise to price manipulation as soon as price impact is nonlinear. "

# Expected price in the two models

Recall from the price process (1) that in our model:

$$\mathbb{E}[S_t] = \int_0^t f(\dot{x}_s)\, G(t-s)\, ds$$

In the AS model, the current spread $D_t$ and the volume impact process $E_t$ are related as

$$D_t = F^{-1}(E_t)$$

so effectively, for continuous trading strategies,

$$\mathbb{E}[S_t] = F^{-1}\left(\int_0^t \dot{x}_s\, e^{-\rho\,(t-s)}\, ds\right)$$

## Cost of trading

The expected cost of trading in the two models is then given by

$$\mathcal{C}_{JG} = \int_0^T \dot{x}_t\, dt \int_0^t f(\dot{x}_s)\, G(t-s)\, ds$$

and

$$\mathcal{C}_{AS} = \int_0^T \dot{x}_t\, dt\, F^{-1}\left(\int_0^t \dot{x}_s\, e^{-\rho\,(t-s)}\, ds\right)$$

These two expression are identical in the OW case with

$$F(x) = x;\ f(v) = v;\ G(t-s) = e^{-\rho\,(t-s)}$$

For more general $F(\cdot)$, the models are different.

# VWAP equivalent models

Are there choices of $F(\cdot)$ for which we can match the expected cost of a VWAP in the two models?

For a VWAP execution, we have $\dot{x}_t = v$, a constant. Then

$$\mathcal{C}_{JG} = v\, f(v) \int_0^T dt \int_0^t G(t-s)\, ds$$

and

$$\mathcal{C}_{AS} = v \int_0^T dt\, F^{-1}\left( v \int_0^t e^{-\rho\,(t-s)}\, ds \right)$$

It turns out that $\mathcal{C}_{JG} = \mathcal{C}_{AS}$ for all $v$ if and only if $F^{-1}(x) = x^{\delta}$, $f(v) = v^{\delta}$ and

$$G(\tau) = \frac{\delta\, \rho\, e^{-\rho\,\tau}}{(1 - e^{-\rho\,\tau})^{1-\delta}}$$

Thus for small $\tau$,

$$G(\tau) \sim \frac{\delta\,\rho}{(\rho\,\tau)^{\gamma}}$$

with $\gamma = 1 - \delta$ and for large $\tau$,

$$G(\tau) \sim \delta\,\rho\,e^{-\rho\,\tau}$$

**Resolution**

With power-law market impact $\propto v^{\delta}$, and exponential resilience, decay of market impact is power-law with exponent $\gamma = 1 - \delta$.

**Remark**

The relationship $\delta + \gamma = 1$ between the exponents emerges naturally from a simple model of the order book!

# Conclusion

- There is no contradiction.
  - Exponential resilience of an order book with power-law shape induces power-law decay of market impact (at least for short times).
- Both AS-style models and processes like (1) are generalizations of Obizhaeva and Wang's model.
  - The AS models are motivated by considerations of dynamics of the order book.
  - The class of AS models with exponential resilience has been shown to be free of price manipulation.
- Our results suggest that the empirically observed relationship $\delta + \gamma \approx 1$ may reflect properties of the order book rather than some self-organizing principle.

# Next steps

- Generalize to non-exponential order book resilience.
- Investigate properties of optimal strategies.
- Derive conditions for no price manipulation in the general case.

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