Diamond trees and the forest expansion

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Outline of this talk

- The diamond product
- The $\mathbb{G}$-expansion
  - Trees and forests
- The $\mathbb{K}$-expansion
  - Third cumulant
- Stochastic volatility and the $\mathbb{F}$-expansion
  - Triple joint MGF
  - The leverage swap
- Explicit computations in affine forward variance models

My last BBQ talk

- In 2018, I presented the diamond product and the exponentiation theorem.
  - Manipulations were formal and the convergence properties of the resulting forest expansion unclear.
- This time we explain the remarkably simple origin of the forest expansion, we give its convergence properties and attempt to give a sense of its wide applicability.

The diamond product

Definition

Given two continuous semimartingales A, B with integrable covariation process $\langle A, B\rangle$, the diamond product[a] of A and B is another continuous semimartingale given by

$$(A \diamond B)_t(T) := \mathbb{E}_t \left[\langle A, B\rangle_{t,T}\right] = \mathbb{E}_t \left[\langle A, B\rangle_T\right] - \langle A, B\rangle_t,$$

where $\langle A, B\rangle_{t,T} = \langle A, B\rangle_T - \langle A, B\rangle_t$.

[a] Warning. Our diamond product is (very) different from the Wick product.

Properties of the diamond product

- Commutative: $A \diamond B = B \diamond A$.
- Non-associative: $(A \diamond B) \diamond C \neq A \diamond (B \diamond C)$.
- $A \diamond B$ depends only on the respective martingale parts of A and B.
- $A \diamond B$ is in general not a martingale.

The $\mathbb{G}$-forest expansion

Theorem 1 (Theorem 1.1 of [FGR20])

Let Y_T be a real-valued, $\mathcal{F}_T$-measurable random variable with associated martingale $Y_t = \mathbb{E}_t[Y_T]$. Under natural integrability conditions, with a, b small enough, there is a.s. convergence of

$$\log \mathbb{E}_t\left[e^{aY_T + b\langle Y\rangle_T}\right] = aY_t + b\langle Y\rangle_t + \sum_{k\geq 2} \mathbb{G}_t^k(T), \tag{1}$$

where

$$\begin{aligned} \mathbb{G}^2 &= \left(\frac{1}{2}a^2 + b\right)(Y \diamond Y)_t(T), \\ \mathbb{G}^k &= \frac{1}{2}\sum_{j=2}^{k-2} \mathbb{G}^{k-j} \diamond \mathbb{G}^j + (a\, Y \diamond \mathbb{G}^{k-1}) \text{ for } k > 2. \end{aligned} \tag{2}$$

Idea of the proof

For a generic (continuous) semimartingale Z, sufficiently integrable, let

$$\Lambda_t^T = \log \mathbb{E}_t \left[e^{Z_{t,T}} \right].$$

Then, noting that $\Lambda_T^T = 0$,

$$\mathbb{E}_t \left[e^{Z_T} \right] = \mathbb{E}_t \left[e^{Z_T + \Lambda_T^T} \right] = e^{Z_t + \Lambda_t^T}.$$

The stochastic logarithm $\mathcal{L}\left(\mathbb{E}_\bullet(Z_T)\right) = Z + \Lambda^T + \frac{1}{2}\langle Z + \Lambda^T \rangle$ is a martingale. Thus,

$$\begin{aligned} \Lambda_t^T &= \mathbb{E}_t \left[Z_{t,T} + \tfrac{1}{2}\langle Z + \Lambda^T \rangle_{t,T} \right] \\ &= \mathbb{E}_t \left[Z_{t,T} \right] + \tfrac{1}{2}((Z + \Lambda^T) \diamond (Z + \Lambda^T))_t(T). \end{aligned}$$

Now with[1] $Z = \epsilon a Y + \epsilon^2 \, b \langle Y \rangle$ we get

$$\Lambda_t^T(\epsilon) = \epsilon a \, \mathbb{E}_t \left[Y_{t,T} \right] + \epsilon^2 \, b \, (Y \diamond Y)_t(T) + \tfrac{1}{2} \left(\epsilon a Y + \Lambda_t^T(\epsilon) \right)_t^{\diamond 2} (T) \, .$$

Put $\Lambda_t^T(\epsilon) = \epsilon^2 \mathbb{G}_t^2 + \epsilon^3 \mathbb{G}_t^3 + \ldots$, and match coefficients of ϵ^n.

$[\epsilon^2]$: $\mathbb{G}_t^2 = b \, (Y \diamond Y)_t(T) + \frac{1}{2} a^2 \, (Y \diamond Y)_t(T)$.

$[\epsilon^3]$: $\mathbb{G}_t^3 = (a \, Y \diamond \mathbb{G}^2)_t(T)$.

$[\epsilon^4]$: $\mathbb{G}_t^4 = (a \, Y \diamond \mathbb{G}^3)_t(T) + \frac{1}{2} (\mathbb{G}^2 \diamond \mathbb{G}^2)_t(T)$.

- We see the recursion (2) emerge!

[1] Recall that terms of bounded variation such as $\langle Y \rangle$ do not contribute to diamond products.

Special cases

Interesting special cases include

- The exponential martingale: $b = -\frac{1}{2}a^2$. All corrector terms $\mathbb{G}^k$ vanish.
- The $\mathbb{F}$-forest expansion of [AGR2020] (working paper 2017): $\frac{1}{2}a + b = 0$.
    - The $\mathbb{F}$-forest expansion gives a general expression for the characteristic function of the log-stock price in a stochastic volatility model written in forward variance form.
- The cumulant ($\mathbb{K}$-forest) expansion of Lacoin-Rhodes-Vargas [LRV19]: $b = 0$.
    - Their expansion was derived in the context of renormalization of the sine-Gordon model in quantum physics.

Further applications

- In [FGR20], a number of other applications are given, including a neat (the neatest?) derivation of the mgf of the Lévy area.
- Other applications likely include
  - computation of likelihood functions in statistics,
  - computation of correlation functions in statistical physics,
  - computation of amplitudes in quantum theory.

- It's beautiful to see that problems in quantitative finance and quantum physics lead to the same mathematics!

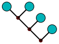
Trees and forests

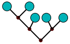
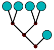
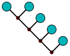
- The general term $\mathbb{G}_t^n(T)$ in (2) is naturally written as a linear combination of binary diamond trees[2].
- Hence the terminology $\mathbb{G}$-*forest expansion* for (1).
- Specifically, writing $\bullet$ as a short-hand for Y, interpreted as single leaf, we have

$$\mathbb{G}^2 = \left(\tfrac{1}{2}a^2 + b\right)$$


$$\mathbb{G}^3 = a\left(\tfrac{1}{2}a^2 + b\right)$$


$$\mathbb{G}^4 = \tfrac{1}{2}\left(\tfrac{1}{2}a^2 + b\right)^2$$

$$+\, a^2\left(\tfrac{1}{2}a^2 + b\right)$$


$$\mathbb{G}^5 = a\left(\tfrac{1}{2}a^2 + b\right)^2$$

$$+\, \tfrac{1}{2}a\left(\tfrac{1}{2}a^2 + b\right)^2$$

$$+\, a^3\left(\tfrac{1}{2}a^2 + b\right)$$

(3)

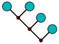
[2] Trees stolen from [Hai13]!

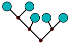
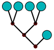
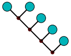
The $\mathbb{K}$-forest expansion

As mentioned earlier, the $\mathbb{K}$-forest expansion ($\mathbb{K}$ for "Kumulant") is obtained by setting $b = 0$ in (1). This gives

$\mathbb{K}^2 = \frac{1}{2} a^2$ 

$\mathbb{K}^3 = \frac{1}{2} a^3$ 

$\mathbb{K}^4 = \frac{1}{8} a^4$  $+ \frac{1}{2} a^4$ 

$\mathbb{K}^5 = \frac{1}{4} a^5$  $+ \frac{1}{8} a^5$  $+ \frac{1}{2} a^5$ 

With $\mathbb{K}^1 =$ , the $\mathbb{K}$-recursion follows naturally.

The $\mathbb{K}$-forest expansion

Theorem 2 (Theorem 1.2 of [FGR20])

Let A_T be $\mathcal{F}_T$-measurable with $N \in \mathbb{N}$ finite moments. Then the recursion

$$\mathbb{K}_t^{n+1}(T) = \frac{1}{2}\sum_{k=1}^{n}(\mathbb{K}^k \diamond \mathbb{K}^{n+1-k})_t(T), \quad \forall n > 0$$

with $\mathbb{K}_t^1(T) := \mathbb{E}_t\left[A_T\right]$ is well-defined up to $\mathbb{K}^N$ and, for $a \in \mathbb{R}$,

$$\log \mathbb{E}_t\left[e^{iaA_T}\right] = \sum_{n=1}^{N}(ia)^n\,\mathbb{K}_t^n(T) + o(|a|^N)$$

which identifies $n! \times \mathbb{K}_t^n(T)$ as the (time t-conditional) n.th cumulant of A_T.

Example: $\mathbb{K}^3$ and the third central moment

- For higher n, the forest expansion encodes relations that are increasingly complex to derive by hand.
- For example, from the forest expansion we have

$$\mathbb{K}^3_t(T) = \tfrac{1}{2}\,(Y \diamond (Y \diamond Y))_t(T)$$

and also, since the third cumulant is the third central moment,

$$\mathbb{K}^3_t(T) = \frac{1}{3!}\,\mathbb{E}_t\left[{Y_{t,T}}^3\right].$$

- On the other hand, the relation

$$\tfrac{1}{2}\,(Y \diamond (Y \diamond Y))_t(T) = \frac{1}{3!}\,\mathbb{E}_t\left[{Y_{t,T}}^3\right]$$

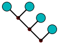
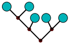
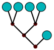
is not so obvious.

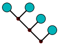
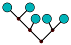
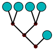
A bivariate $\mathbb{K}$-expansion

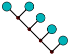
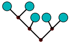
Let $\mathbb{K}^1_t = \mathbb{E}_t\left[a\, Y_T + b\,\langle Y\rangle_{t,T}\right] \equiv a$  $+\, b$ . Then

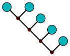
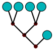
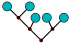
$\mathbb{K}^1 \;=\; a$  $+\, b$ 

$\mathbb{K}^2 \;=\; \frac{1}{2}\,(a$  $+\, b$  $)^{\diamond 2} = \frac{1}{2} a^2$  $+\, ab$  $+\, \frac{1}{2} b^2$ 

$\mathbb{K}^3 \;=\; \frac{1}{2} a^3$  $+\, \frac{1}{2} a^2 b$  $+\, a^2 b$  $+\, ab^2$  $+\, \frac{1}{2} ab^2$  $+\, \ldots$

$\mathbb{K}^4 \;=\; \frac{1}{2} a^4$  $+\, \frac{1}{2^3} a^4$  $+\, \frac{1}{2} a^3 b$  $+\, \frac{1}{2} a^3 b$ 

$+\, a^3 b$  $+\, \frac{1}{2} a^3 b$  $+\, \ldots$

$\mathbb{K}^5 \;=\; \frac{1}{2} a^5$  $+\, \frac{1}{2^3} a^5$  $+\, \frac{1}{2^2} a^5$  $+\, \ldots \qquad (4)$

Forest reordering

- We see that the $\mathbb{G}$-recursion is equivalent to the bivariate $\mathbb{K}$-recursion applied to $A_T = aY_T + b\langle Y\rangle_T$, after forest reordering.
    - Reorder by collecting all trees with the same number of leaves.
    - $\mathbb{G}$-forests consist of trees which are homogenous in the number of leaves $\bullet$ but not in a, b.
- Note also that forest reordering resolves the infinite cancellations present in the bivariate $\mathbb{K}$-expansion.
    - To see this put $b = -\frac{1}{2}a^2$ in (4) – we see a very complicated expression which must sum to zero.
    - On the other hand putting $b = -\frac{1}{2}a^2$ in (3) trivially results in zero.

Forward variance models

- Let S be a strictly positive continuous martingale.
- Then $X := \log S$ is a semimartingale with quadratic variation process $\langle X \rangle$.
- Following [BG12], it is natural to specify a model in forward variance form.

$$
\begin{aligned}
v_t \, dt &:= d\langle X \rangle_t \\
\xi_t(T) &= \mathbb{E}_t \left[v_T \right].
\end{aligned}
$$

  - Forward variances are tradable assets (unlike spot variance).
  - We get a family of martingales indexed by their individual time horizons T.

VIX squared

- Consider the payoff of a forward-starting variance swap

$$\begin{aligned}\zeta_T(T) &= \frac{1}{\Delta}\int_T^{T+\Delta} \xi_T(u)\,du \\ &= \frac{1}{\Delta}\int_T^{T+\Delta} \mathbb{E}_T \int_T^{T+\Delta} v_u\,du \\ &= \frac{1}{\Delta}\int_T^{T+\Delta} \mathbb{E}_T \langle X \rangle_{T,T+\Delta},\end{aligned}$$

which, when Δ is 30 days, is just VIX squared.

- The $\mathbb{G}$-expansion gives us the joint MGF of VIX^2, X and $\langle X \rangle$ as follows.

Triple joint MGF

Theorem 3 (Theorem 4.4 of [FGR20])

For $a, b, c \in \mathbb{R}$ sufficiently small,

$$\mathbb{E}_t\left[e^{a\,X_T + b\,\langle X\rangle_{t,T} + c\,\zeta_T(T)}\right] = \exp\left\{a\,X_t + c\,\zeta_t(T) + \sum_{k=2}^{\infty}\mathbb{G}_t^k\right\},$$

where

$$\begin{aligned}\mathbb{G}^2 &= \left(\tfrac{1}{2}a\,(a-1)+b\right)(X\diamond X)_t(T) + ac\,X\diamond\zeta + \tfrac{1}{2}c^2\,\zeta\diamond\zeta,\\ \mathbb{G}^k &= \frac{1}{2}\sum_{j=2}^{k-2}\mathbb{G}^{k-j}\diamond\mathbb{G}^j + (a\,X\diamond\mathbb{G}^{k-1}) \text{ for } k>2.\end{aligned}$$

Proof.

This is a direct consequence of Theorem 1: The time-T quantity of interest is

$$A_T := a\,X_T + b\,\langle X\rangle_{t,T} + c\,\zeta_T(T)$$

and it suffices to compute (using that $X + \frac{1}{2}\langle X\rangle$ is martingale),

$$\mathbb{E}_t\left[A_T\right] = a\,X_t + (b - \tfrac{1}{2}\,a)\,(X \diamond X)_t(T) + c\,\zeta_t(T)\,.$$

$\square$

- Theorem 3 is completely *model-independent*!
    - It is useful in particular when the diamond trees are easy to compute or approximate.
- We can get the joint MGF of any set of random variables of interest in the same way.
    - For example, VIX futures are martingales. So the joint MGF of SPX and VIX is in principle computable!

Trees with colored leaves

Denote $X \equiv \bullet$ and $\zeta \equiv \bullet$.

- In Theorem 3 we wrote

$$\mathbb{G}^2 = \left(\tfrac{1}{2}\,a\,(a-1) + b\right)$$  $$+\, ac$$  $$+\, \tfrac{1}{2}\,c^2$$  .

- We could define $(X \diamond X) = M$, or  $= \bullet$, resulting in trees with leaves of three different colors.
    - In a forward variance model, X_t represents the log-stock price and $M_t(T)$, the expected total variance $\int_t^T \xi_t(u)\,du$.
- Then

$$\mathbb{G}^2 = \left(\tfrac{1}{2}\,a\,(a-1) + b\right) \bullet + ac$$  $$+\, \tfrac{1}{2}\,c^2$$  .

- In general, we can always identify subtrees in this way and assign them a new variable name (and leaf color).

$\mathbb{F}$-recursion

Putting $b = -\frac{1}{2}a$ in the $\mathbb{G}$-recursion gives the $\mathbb{F}$-recursion.

Theorem 4

With $\mathbb{F}^2 = \frac{1}{2}a\,(a-1) = \frac{1}{2}a\,(a-1)\bullet$ *and* $\forall k > 2$,

$$\mathbb{F}^k = \frac{1}{2}\sum_{j=2}^{k-2} \mathbb{F}^{k-j} \diamond \mathbb{F}^j + (a\,Y \diamond \mathbb{F}^{k-1}), \tag{5}$$

and we have, for sufficiently small a,

$$\log \mathbb{E}_t\left[e^{aX_T}\right] = a\,X_t + \sum_{k\geq 2} \mathbb{F}^k\,. \tag{6}$$

On the other hand, Corollary 3.1 of [AGR2020] reads:

Corollary

The cumulant generating function (CGF) is given by

$$\psi_t(T;a) = \log \mathbb{E}_t\left[e^{i\,a\,X_T}\right] = i\,a\,X_t - \frac{1}{2}a\,(a+i)\,M_t(T) + \sum_{\ell=1}^{\infty} \tilde{\mathbb{F}}_\ell(a). \tag{7}$$

where the $\tilde{\mathbb{F}}_\ell$ satisfy the recursion

$\tilde{\mathbb{F}}_0 = -\frac{1}{2}a(a+\mathrm{i})\,M_t = -\frac{1}{2}a(a+\mathrm{i}) \bullet$ and for $k>0$,

$$\tilde{\mathbb{F}}_\ell = \frac{1}{2}\sum_{j=0}^{\ell-2}\left(\tilde{\mathbb{F}}_{\ell-2-j} \diamond \tilde{\mathbb{F}}_j\right) + \mathrm{i}a\left(X \diamond \tilde{\mathbb{F}}_{\ell-1}\right). \tag{8}$$

- With the identification $\tilde{\mathbb{F}}_\ell = \mathbb{F}^{\ell+2}$, formulae (6) and (7), and the recursions (5) and (8) are equivalent.

Applying the recursion (8), the first few $\tilde{\mathbb{F}}$ forests are given by

$$
\begin{aligned}
\tilde{\mathbb{F}}_0 &= -\tfrac{1}{2} a(a+\mathrm{i}) \\
\tilde{\mathbb{F}}_1 &= -\tfrac{\mathrm{i}}{2} a^2 (a+\mathrm{i}) \\
\tilde{\mathbb{F}}_2 &= \tfrac{1}{2^3} a^2 \,(a+\mathrm{i})^2 + \tfrac{1}{2} a^3 \,(a+\mathrm{i}) \\
\tilde{\mathbb{F}}_3 &= (\tilde{\mathbb{F}}_0 \diamond \tilde{\mathbb{F}}_1) + \mathrm{i}a \bullet \diamond \tilde{\mathbb{F}}_2 \\
&= \tfrac{\mathrm{i}}{2^2} a^3 \,(a+\mathrm{i})^2 + \tfrac{\mathrm{i}}{2^3} a^3 \,(a+\mathrm{i})^2 + \tfrac{\mathrm{i}}{2} a^4 \,(a+\mathrm{i}) .
\end{aligned}
$$

- Note that the total probability and martingale constraints are satisfied for each tree.
  - That is $\psi_t^T(0) = \psi_t^T(-\mathrm{i}) = 0$.

Variance and gamma swaps

The variance swap is given by the fair value of the log-strip:

$$\mathbb{E}_t\left[X_T\right] = (-\mathrm{i})\,\psi_t^{T\prime}(0) = X_t - \tfrac{1}{2}\,M_t(T)$$

and the gamma swap (wlog set $X_t = 0$) by

$$\mathbb{E}_t\left[X_T\, e^{X_T}\right] = -\mathrm{i}\,\psi_t^{T\prime}(-\mathrm{i}).$$

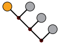
Remark

We can in principle compute such moments for any stochastic volatility model written in forward variance form, whether or not there exists a closed-form expression for the characteristic function.

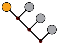
The gamma swap

It is easy to see that only trees containing a single $\bullet$ leaf will survive in the sum after differentiation when $a = -\mathrm{i}$ so that

$$\sum_{\ell=1}^{\infty} \tilde{\mathbb{F}}'_{\ell}(-\mathrm{i}) = \frac{\mathrm{i}}{2} \sum_{\ell=1}^{\infty} X^{\diamond \ell} M$$

$= \frac{\mathrm{i}}{2} \Big\{$  $+$  $+$  $+ \ldots \Big\}$

Then the fair value of a gamma swap is given by

$\mathcal{G}_t(T) = 2\,\mathbb{E}_t\left[X_T\, e^{X_T}\right] = \bullet +$  $+$  $+$  $+ \ldots \qquad (9)$

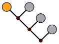
Remark

Equation (9) allows for explicit computation of the gamma swap for any model written in forward variance form.

The leverage swap

We deduce that the fair value of a leverage swap is given by

$$\mathcal{L}_t(T) = \mathcal{G}_t(T) - M_t(T) = \sum_{\ell=1}^{\infty} X^{\diamond \ell} M$$

$=$  $+$  $+$  $+ \ldots$ (10)

- The leverage swap is expressed explicitly in terms of covariance products of the spot and vol. processes.
  - If spot and vol. processes are uncorrelated, the fair value of the leverage swap is zero.

An explicit model-free expression for the leverage swap!

$\mathcal{L}_t(T)$ directly from the smile

- Let
$$d_\pm(k) = \frac{-k}{\sigma_{\rm BS}(k,T)\sqrt{T}} \pm \frac{\sigma_{\rm BS}(k,T)\sqrt{T}}{2}$$
and following Fukasawa [Fuk12], denote the inverse functions by $g_\pm(z) = d_\pm^{-1}(z)$. Further define
$$\sigma_\pm(z) = \sigma_{\rm BS}(g_\pm(z),T)\,\sqrt{T}.$$
- It is a well-known corollary of Matytsin's characteristic function representation in [Mat00], that
$$M_t(T) = \int_{\mathbb{R}} dz\, N'(z)\,\sigma_-^2(z).$$
- The gamma swap is given by
$$\mathcal{G}_t(T) = \int_{\mathbb{R}} dz\, N'(z)\,\sigma_+^2(z).$$

Fast calibration

- For each T, $\mathcal{L}_t(T) = \mathcal{G}_t(T) - M_t(T)$ may be estimated from the observed smile.
  - In the case of SPX, there are currently between 30 and 40 listed expirations.
- Also, $\mathcal{L}_t(T) = \sum_{\ell=1}^{\infty} X^{\diamond \ell} M$.
- For models (such as affine forward variance models) where diamond trees are easily computable, fast calibration is then possible.

Affine forward variance models

Following [GKR19] consider *affine forward variance models* of the form

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{v_t}\, dZ_t \\ d\xi_t(u) &= \kappa(u-t)\,\sqrt{v_t}\, dW_t, \end{aligned}$$

with $d\langle W, Z\rangle_t = \rho\, dt$.

- This class of models includes classical and rough Heston.
- As we will see, diamond trees are particularly easy to compute in AFV models.

Affine trees

Lemma 5 (Lemma 4.5 of [FGR20])

In an affine forward variance model, all diamond trees take the form

$$\int_t^T \xi_t(u)\, h(T-u)\, du$$

for some function h.

Classical Heston

Example (Classical Heston)

In this case,

$$d\xi_t(u) = \nu\, e^{-\lambda\,(u-t)}\, \sqrt{v_t}\, dW_t.$$

Then, for example,

$$= (X \diamond M)_t(T) \quad = \quad \frac{\rho\,\nu}{\lambda} \int_t^T \xi_t(u) \left[1 - e^{-\lambda(T-u)}\right] du.$$

Rough Heston

Example (Rough Heston)

In this case, with $\alpha = H + 1/2 \in (1/2, 1)$ (and with $\lambda = 0$),

$$d\xi_t(u) = \frac{\nu}{\Gamma(\alpha)} (u-t)^{\alpha-1} \sqrt{v_t}\, dW_t.$$

Then, for example,

$$\bullet = M_t(T) = (X \diamond X)_t(T) = \int_t^T \xi_t(u)\, du,$$

$$= \frac{\nu^2}{\Gamma(\alpha)^2} \int_t^T \xi_t(u)\, du \left(\int_u^T (s-u)^{\alpha-1}\, ds \right)^2$$

$$= \frac{\nu^2}{\Gamma(1+\alpha)^2} \int_t^T \xi_t(u)\, (T-u)^{2\alpha}\, du.$$

- For a bounded forward variance curve ξ one then sees that diamond trees with k leaves are of order $(T-t)^{1+(k-2)\alpha}$.
- In this case, the $\mathbb{F}$-expansion (forest reordering according to number of leaves) has the interpretation of a short-time expansion, the concrete powers of which depend on the roughness parameter $\alpha = H + 1/2 \in (1/2, 1)$, cf. [CGP18, GR19].

The triple joint MGF in affine forward variance models

- Lemma 5 combined with Theorem 3 characterize the triple-joint MGF of X_T, $\langle X \rangle_T$ and $\zeta_T(T)$ for an affine forward variance model.
  - Compare with Theorem 4.3 of [AJLP2019] and Proposition 4.6 of [GKR19].
- We obtain the convolutional form

$$\mathbb{E}_t\left[e^{a\,X_T + b\,\langle X\rangle_{t,T} + c\,\zeta_T(T)}\right] = \exp\left\{a\,X_t + (\xi \star g)(\tau; a, b, c)_t(T)\right\} \;.$$

- This is consistent with (and generalizes) Theorem 2.6 of [GKR19] where the same convolution Riccati equation appears, but with $g = g(\tau; a)$ instead of $(\tau; a, b, c)$ and different boundary conditions.

Computation of trees under rough Heston

Abbreviating bounded variation terms as 'BV', we have

$$
\begin{aligned}
dX_t &= \sqrt{v_t}\, dZ_t + \mathrm{BV} \\
dM_t &= \int_t^T d\xi_t(u)\, du + \mathrm{BV} \\
&= \frac{\nu}{\Gamma(\alpha)} \sqrt{v_t} \left(\int_t^T \frac{du}{(u-t)^\gamma} \right) dW_t + \mathrm{BV} \\
&= \frac{\nu\,(T-t)^\alpha}{\Gamma(1+\alpha)} \sqrt{v_t}\, dW_t + \mathrm{BV}\,.
\end{aligned}
$$

The first order forest

There is only one tree in the forest $\tilde{\mathbb{F}}_1$.

$$\tilde{\mathbb{F}}_1 = $$  $$ = (X \diamond M)_t(T) \;=\; \mathbb{E}_t\left[\int_t^T d\langle X, M\rangle_s\right]$$

$$= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\,\mathbb{E}_t\left[\int_t^T v_s\,(T-s)^\alpha\,ds\right]$$

$$= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\int_t^T \xi_t(s)\,(T-s)^\alpha\,ds.$$

Higher order forests

Define for $j \geq 0$

$$I_t^{(j)}(T) := \int_t^T ds\, \xi_t(s)\, (T-s)^{j\,\alpha}.$$

Then

$$\begin{aligned} dI_s^{(j)}(T) &= \int_s^T du\, d\xi_s(u)\, (T-u)^{j\,\alpha} + \mathrm{BV} \\ &= \frac{\nu\sqrt{v_s}}{\Gamma(\alpha)}\, dW_s \int_s^T \frac{(T-u)^{j\,\alpha}}{(u-s)^{\gamma}}\, du + \mathrm{BV} \\ &= \frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)}\, \nu\sqrt{v_s}\, (T-s)^{(j+1)\,\alpha}\, dW_s + \mathrm{BV}. \end{aligned}$$

With this notation,

 $$= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\, I_t^{(1)}(T).$$

The second order forest

There are two trees in $\tilde{\mathbb{F}}_2$:

$$
\begin{aligned}
&= \mathbb{E}_t \left[\int_t^T d\langle M, M\rangle_s \right] \\
&= \frac{\nu^2}{\Gamma(1+\alpha)^2} \int_t^T \xi_t(s)\,(T-s)^{2\,\alpha}\,ds \\
&= \frac{\nu^2}{\Gamma(1+\alpha)^2}\, I_t^{(2)}(T)
\end{aligned}
$$

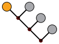
and

$$
\begin{aligned}
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\, \mathbb{E}_t \left[\int_t^T d\langle X, I^{(1)}\rangle_s \right] \\
&= \frac{\rho^2\,\nu^2}{\Gamma(1+2\,\alpha)}\, I_t^{(2)}(T).
\end{aligned}
$$

The third order forest

Continuing to the forest $\tilde{\mathbb{F}}_3$, we have the following.

$$= \frac{\rho\,\nu^3}{\Gamma(1+\alpha)\,\Gamma(1+2\,\alpha)}\,I_t^{(3)}(T)$$



$$= \frac{\rho^3\,\nu^3}{\Gamma(1+3\,\alpha)}\,I_t^{(3)}(T)$$

$$= \frac{\rho\,\nu^3\,\Gamma(1+2\,\alpha)}{\Gamma(1+\alpha)^2\,\Gamma(1+3\,\alpha)}\,I_t^{(3)}(T).$$

In particular, we readily identify the pattern

$$\left(X^{\diamond\ell} M\right)_t(T) = \frac{(\rho\,\nu)^\ell}{\Gamma(1+\ell\,\alpha)}\,I_t^{(\ell)}(T).$$

The leverage swap under rough Heston

Using (10), we have

$$
\begin{aligned}
\mathcal{L}_t(T) &= \sum_{\ell=1}^{\infty} \left(X^{\diamond \ell} M\right)_t (T) \\
&= \sum_{\ell=1}^{\infty} \frac{(\rho\,\nu)^\ell}{\Gamma(1+\ell\,\alpha)} \int_t^T du\, \xi_t(u)\,(T-u)^{\ell\,\alpha} \\
&= \int_t^T du\, \xi_t(u)\, \{E_\alpha(\rho\,\nu\,(T-u)^\alpha) - 1\}
\end{aligned}
$$

where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function.

An explicit expression for the leverage swap!

- Since we can impute the leverage swap $\mathcal{L}_t(t)$ from the smile for each expiration T, fast calibration of the rough Heston model is possible.

Summary

- We introduced the diamond product.
- We defined the $\mathbb{G}$-expansion and gave an idea of its proof.
  - The cumulant expansion of [LRV19] and the Exponentiation Theorem of [AGR2020] are special cases.
- We showed how easy computations can be in affine forward variance models.
  - Quick calibration of such models is one application.

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