The Variational Most-Likely-Path

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Outline

- Local volatility in terms of implied volatility
- Implied volatility in terms of local volatility
- Small-T approximations
- The most-likely-path idea
- The variational most-likely-path
- Solving for the vMLP
- Numerical tests with a realistic volatility surface

Objective

Given a local volatility process

$$\frac{dS}{S} = \sigma_\ell(S,t)\, dW_t,$$

with $\sigma_\ell(S,t)$ depending only on the underlying level S and the time t, we want to compute implied volatilities $\sigma_{BS}(K,T)$ such that

$$C_{BS}(S,K,\sigma_{BS}(K,T),T) = \mathbb{E}\left[(S_T - K)^+\right]$$

or in words, we want to efficiently compute implied volatility from local volatility.

- This can of course be done with numerical PDE
  - but numerical PDE is slow,
  - too slow for efficient calibration to implied volatilities.

Motivations

- The condition for no static arbitrage can be simply expressed as the non-negativity of local variance.
  - It's very hard in general to eliminate static arbitrage in a given parameterization of the implied volatility surface.
- Knowing how to get implied volatility from local volatility helps us get accurate approximations to implied volatility in more complex models such as SABR.
  - Efficient calibration of complex models becomes practical.

Local volatility in terms of implied volatility

Define the Black-Scholes implied total variance:

$$w(K,T) := \sigma_{BS}^2(K,T)\;T$$

In terms of the log-strike $k := \log K/F$ and the local variance $v_\ell := \sigma_\ell^2(K,T)$, the Dupire equation becomes

$$\frac{\partial C}{\partial T} = \frac{v_\ell}{2}\left\{\frac{\partial^2 C}{\partial k^2} - \frac{\partial C}{\partial k}\right\}$$

Then, by taking derivatives of the Black-Scholes formula and simplifying, we obtain equation (1.10) in [The Volatility Surface]:

$$v_\ell = \frac{\frac{\partial w}{\partial T}}{\left(1 - \frac{k}{2\,w}\,\frac{\partial w}{\partial k}\right)^2 - \frac{1}{4}\,\left(\frac{1}{4} + \frac{1}{w}\right)\,\left(\frac{\partial w}{\partial k}\right)^2 + \frac{1}{2}\,\frac{\partial^2 w}{\partial k^2}} \tag{1}$$

Special Case: No Skew

If the skew $\frac{\partial w}{\partial k}$ is zero, (1) reduces to

$$v_\ell = \frac{\partial w}{\partial T}$$

In this special case, the local variance reduces to the forward Black-Scholes implied variance. The solution is of course

$$w\,(T) = \int_0^T v_\ell\,(t)\;dt$$

or equivalently in the suggestive form,

$$\sigma_{BS}^2(T) = \frac{1}{T}\int_0^T \sigma_\ell^2(t)\,dt = \int_0^1 \sigma_\ell^2(\alpha\,T)\,d\alpha.$$

Inverting the equation

- We have a formula (1) for getting local volatility from implied.
- All we need to do is to invert this formula!
  - This is certainly not easy and has not so far proved to be possible in closed-form.
- In the limit of small time however, equation (1) can been solved.

The BBF approximation

Recall equation (1) for local variance in terms of implied:

$$v_\ell = \frac{\frac{\partial w}{\partial T}}{\left(1 - \frac{k}{2\,w}\,\frac{\partial w}{\partial k}\right)^2 - \frac{1}{4}\,\left(\frac{1}{4} + \frac{1}{w}\right)\,\left(\frac{\partial w}{\partial k}\right)^2 + \frac{1}{2}\,\frac{\partial^2 w}{\partial k^2}}$$

Noting that $w \sim O(T)$, in the limit of small T, to leading order in T we may write

$$v_\ell \approx \frac{\frac{\partial w}{\partial T}}{\left(1 - \frac{k}{2\,w}\,\frac{\partial w}{\partial k}\right)^2} \tag{2}$$

Further supposing that to lowest order in T, $w \approx \sigma_{BS}(k, 0)^2\,T$ and making the change of variable

$$u = \frac{1}{\sigma_{BS}(k, 0)},$$

we may rewrite (2) as

$$\sigma(k,0)^2 \approx \frac{\frac{1}{u^2}}{\left(1+\frac{k}{u}\frac{\partial u}{\partial k}\right)^2}$$

or rearranging

$$\frac{\partial}{\partial k}(k\,u) = \frac{1}{\sigma(k,0)}$$

giving us the BBF approximation of [Berestycki, Busca and Florent]:

$$\frac{1}{\sigma_{BS}(K,T)} \approx \frac{1}{\sigma_0(k)} := \frac{1}{\ln K/S_0}\int_{S_0}^{K}\frac{dS}{S\,\sigma(S,0)} = \int_0^1 \frac{d\alpha}{\sigma(\alpha\,k,0)}$$

First order term

[Henry-Labordère] expands $\sigma_{BS}(\cdot)$ as

$$\sigma_{BS}(k, T) = \sigma_0(k) + \sigma_1(k)\,T + O(T^2).$$

Substituting into (1) and matching powers of T, he obtains the first order correction:

$$\begin{aligned}\sigma_1(k) \quad &= \quad \frac{\sigma_0(k)^3}{k^2}\left\{\ln\frac{\sqrt{\sigma(0,0)\,\sigma(k,0)}}{\sigma_0(k)}\right.\\ &\qquad \left. - \int_0^k \frac{\partial_t\sigma(y,t)|_{t=0}}{\sigma(y,0)}\,\frac{\partial}{\partial y}\left(\frac{y}{\sigma_0(y)}\right)^2 dy\right\}\end{aligned}$$

where $\sigma_0(k)$ is the lowest-order (BBF) approximation derived earlier.

Heat kernel expansion

In [GHLOW], we compute implied volatility for short times using the heat kernel expansion up to second order.

$$\sigma_{BS}(k, T) \approx \sigma_0(k) + \sigma_1(k)\, T + \sigma_2(k)\, T^2$$

The first two terms, σ_0 and σ_1 agree with BBF and H-L respectively. σ_2 is somewhat too complicated to reproduce here!

Call price in terms of the transition density

Let $p(t, s; t', s')$ be the transition probability density. Then

$$\begin{aligned} C(s, t, K, T) &= \mathbb{E}\left[(S_T - K)^+ | S_t = s\right] \\ &= \int (s' - K)^+ p(t, s; T, s') ds' \end{aligned} \tag{3}$$

As a function of t and s, p satisfies the backward Kolmogorov equation:

$$\mathcal{L}p := p_t + \frac{1}{2}\, s^2\, \sigma_\ell^2(s, t)\, p_{ss} = 0,$$

Subindices refer to respective partial derivatives.

Heat kernel expansion

Heat kernel expansion for transition density $p(t, s; t', s')$ when $t' - t$ is small:

$$p(t, s; t', s') \sim \frac{e^{-\frac{d^2(s,s',t)}{2(t'-t)}}}{\sqrt{2\pi(t'-t)}s'\sigma_\ell(s', t')} \sum_{k=0}^{n} H_k(t, s, s')(t' - t)^k$$

- $d(s, s', t) = \left| \int_s^{s'} \frac{d\xi}{\xi\sigma_\ell(\xi,t)} \right|$: geodesic distance between s to s'
- $H_0(t, s, s') = \sqrt{\frac{s\sigma_\ell(s,t)}{s'\sigma_\ell(s',t)}} \exp\left[\int_s^{s'} \frac{d_t(\eta,s',t)}{\eta\sigma_\ell(\eta,t)} d\eta \right]$
- $H_i(t, s, s') = \frac{H_0(t,s,s')}{d^i(s,s',t)} \int_{s'}^{s} \frac{d^{i-1}(\eta,s',t)LH_{i-1}}{H_0(\eta,s',t)a(\eta,t)} d\eta$

Heat kernel expansion for Black-Scholes

Heat kernel expansion for Black-Scholes transition density $p_{BS}(t,s;t',s')$ when $t'-t$ is small:

$$p_{BS}(t'-t,s,s') = \frac{e^{-\frac{d_{BS}^2(s,s')}{2(t'-t)}}}{\sqrt{2\pi(t'-t)}\,\sigma_{BS}\,s'}\sqrt{\frac{s}{s'}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[\frac{\sigma_{BS}^2(t'-t)}{8}\right]^k$$

- $d_{BS}(s,s') = \left|\int_s^{s'} \frac{d\xi}{\sigma_{BS}\xi}\right| = \frac{1}{\sigma_{BS}}\left|\log\frac{s'}{s}\right|$
- The lowest order heat kernel coefficient is given by

$$H_0^{BS}(t,s,s') = \sqrt{\frac{s}{s'}}.$$

The heat-kernel expansion of implied volatility

Implied volatility σ_{BS} is defined as the unique solution to

$$C(s,t,K,T) = C_{BS}(s,t,K,T,\sigma_{BS}) \tag{4}$$

- Substitute the heat kernel approximation to the transition density $p(\cdot)$ into the expression for both the model price C and the Black-Scholes price C_{BS}.
- Expand both sides of (4) in powers of $T-t$.
- On the RHS, also expand the BS implied volatility:

$$\sigma_{BS}(K,T) \approx \sigma_{BS,0} + \sigma_{BS,1}\,(T-t) + \sigma_{BS,2}\,(T-t)^2$$

- Match the coefficients of powers of $T-t$.

What's wrong with small-time expansions?

- Small-time expansions like that of [GHLOW] use only information about the volatility surface in a neighborhood of the origin (the at-the-money, zero time to expiration point).
- Shouldn't an implied volatility approximation take into account all of the local volatility surface?
  - Information from the neighborhood of the origin does generate accurate implied volatility estimates for time-homogeneous models such as CEV.
  - This cannot be the case for empirically reasonable local volatility surfaces.
  - For empirically reasonable local volatility surfaces, the k and t derivatives may not even exist at the origin in which case the short-time expansions presented earlier cannot be performed.

Integral representation of implied volatility

Alternatively, we have integral representations of implied volatility such as the one presented in [The Volatility Surface]:

$$\sigma_{BS}(K,T)^2 = \bar{\sigma}(0)^2 = \frac{1}{T}\int_0^T \frac{\mathbb{E}\left[\sigma_t^2 S_t^2 \Gamma_{BS}(S_t)\right]}{\mathbb{E}\left[S_t^2 \Gamma_{BS}(S_t)\right]} dt \qquad (5)$$

which expresses implied variance as the time-integral of expected instantaneous variance σ_t^2 under some probability measure.

- Note that equation (5) is circular because the gamma $\Gamma_{BS}(S_t)$ of the option on the rhs depends on $\sigma_{BS}(K,T)$ on the lhs.

Special case: Black-Scholes

Suppose $\sigma_t = \sigma(t)$, a function of t only. Then

$$\frac{\mathbb{E}\left[\sigma(t)^2 S_t^2 \Gamma_{BS}(S_t)\right]}{\mathbb{E}\left[S_t^2 \Gamma_{BS}(S_t)\right]} = \sigma(t)^2$$

and from (5),

$$\sigma_{BS}(K,T)^2 = \frac{1}{T}\int_0^T \sigma^2(t)\,dt$$

which has no dependence on the strike K.

Visualizing implied volatility

Equation (5) may be rewritten in the form

$$\sigma_{BS}^2(K,T) = \frac{1}{T}\int_0^T\int q(S_t; S_0, K, T)\,\sigma_\ell^2(S_t,t)\,dS_t\,dt$$

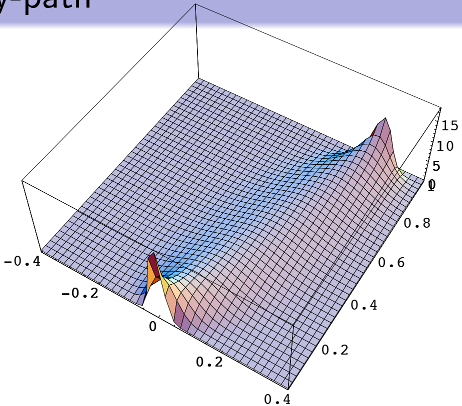
where

$$q\left(S_t, t; S_0, K, T\right) := \frac{p(0, s_0; S_t, t) S_t^2 \Gamma_{BS}(S_t)}{\mathbb{E}\left[S_t^2\,\Gamma_{BS}(S_t)\right]}$$

and $\sigma_\ell^2(S_t,t) = \mathbb{E}^P\left[\sigma_t^2|S_t\right]$ is the local variance or alternatively in terms of $x_t := \log\left(S_t/F\right)$:

$$\sigma_{BS}^2(K,T) = \frac{1}{T}\int_0^T\int q(x_t, t; x_T, T)\sigma_\ell^2(x_t,t)dx_t dt \qquad (6)$$

The most-likely-path



$$\sigma_{BS}^2(K,T) = \frac{1}{T}\int_0^T \int q(x_t,t;x_T,T)\,\sigma_\ell^2(x_t,t)dx_t dt \approx \frac{1}{T}\int_0^T \sigma_\ell^2(\tilde{x}_t,t)dt \quad (7)$$

where $\tilde{x}_t$ is the most likely path.

The most-likely-path formula in words

- Equation (7) says that the Black-Scholes implied variance of an option with strike K is given approximately by the integral from valuation date ($t = 0$) to the expiration date ($t = T$) of the local variances along the most-likely-path.
  - Implied volatility is root-mean-squared local volatility.
- However, not only is it not trivial to compute the path $\tilde{x}_t$ but there is no unique definition of $\tilde{x}_t$.
  - [The Volatility Surface] chooses $\tilde{x}_t$ as the path that maximizes the density $q(x_t, t; x_T, T)$.
  - [Reghai] chooses $\tilde{x}_t$ to be the conditional expectation of x_t wrt $q(\cdot)$.

The main idea: Heat kernel + Chapman Kolmogorov

- We have highly accurate approximations for small T.
- We want to take information about the entire local volatility surface into account.
- Use the heat kernel approximation to approximate the transition density for a small timestep T/n and apply Chapman-Kolmogorov to get an approximation to the density over a large timestep.
- Use the Laplace asymptotic formula to approximate the resulting n-dimensional integral.

Laplace asymptotic formula

Asymptotic expansion of the integral as $\tau \to 0^+$

$$\int_0^\infty e^{-\frac{\phi(x)}{\tau}} f(x)dx \sim \tau^2 e^{-\frac{\phi(x^*)}{\tau}} \frac{f'(x^*)}{[\phi'(x^*)]^2}$$

Assumptions:

- f is identically zero when $0 \leq x \leq x^*$.
- ϕ is increasing in $[x^*, \infty)$.

Example: Approximation to the price of a call

Let $\tau = T - t$. Then

$$
\begin{aligned}
C(s,t,K,T) &= \int_0^\infty (s-K)^+ p(t,s;T,s')ds' \\
&\approx \frac{1}{\sqrt{2\pi\tau}} \int_0^\infty (s'-K)^+ \frac{e^{-\frac{d^2(s,s',t)}{2\tau}}}{s'\,\sigma(s',T)} H_0(t,s,s')ds' \\
&= \frac{1}{\sqrt{2\pi\tau}} \int_K^\infty e^{-\frac{d^2(s,s',t)}{2\tau}} G_0(t,s,T,s')ds' \qquad (8)
\end{aligned}
$$

where

$$G_0(t,s,T,s') = (s'-K)\,\frac{H_0(t,s,s')}{s'\sigma(s',T)}$$

Approximations to the call price

Assuming $s < K$ and performing the integration in (8), we obtain

$$C(s,t,K,T) \approx \frac{\tau^{\frac{3}{2}}}{\sqrt{2\pi}} e^{-\frac{d^2}{2\tau}} \frac{G_0'}{(dd')^2}.$$

- $d = d(s,K,t)$ and $d' = \frac{\partial d}{\partial s'}(s,K,t)$
- $G_0' = \frac{\partial G_0}{\partial s'}(t,s,T,K) = \frac{H_0(t,s,K)}{K\sigma_\ell(K,T)}$

In the special case of Black-Scholes (with $k = \log K/s$), we get

$$C_{BS}(s,t,K,T,\sigma_{BS}) \approx \sqrt{s\,K}\,\frac{\tau^{\frac{3}{2}}}{\sqrt{2\pi}} e^{-\frac{k^2}{2\sigma_{BS}^2\tau}} \frac{\sigma_{BS}^3}{k^2}.$$

Chapman Kolmogorov

Let $\mathbf{s} = (s_0, \cdots, s_n)$ and $\mathbf{t} = (t_0, \cdots, t_n)$. By Chapman-Kolmogorov we have

$$p(s_0, t_0; s_n, t_n) = \int \prod_{k=1}^{n} p(s_{k-1}, t_{k-1}; s_k, t_k) d\mathbf{s}.$$

The option price then becomes an n dimensional integration

$$\begin{aligned} \mathbb{E}[(S_T - K)^+] &= \int (s_n - K)^+ \prod_{i=1}^{n} p(t_{i-1}, s_{i-1}; t_i, s_i) d\mathbf{s} \\ &= \int_{s_n \geq K} (s_n - K) \prod_{i=1}^{n} p(t_{i-1}, s_{i-1}; t_i, s_i) d\mathbf{s} \end{aligned}$$

Recap of steps in derivation of MLP

$$\mathbb{E}[(S_T - K)^+] = \int_{s_n \geq K} (s_n - K) \prod_{i=1}^{n} p(t_{i-1}, s_{i-1}; t_i, s_i) d\mathbf{s}$$

- Express the transition density as a convolution of transition densities over small time intervals using Chapman-Kolmogorov.
- Approximate transition density by the heat kernel expansion over each small time interval.
- Approximate the resulting n-dimensional integral using the Laplace asymptotic formula.
- Push n to infinity.

Heat kernel expansion again

Heat kernel expansion for transition density $p(t, s; t', s')$ when $t' - t$ is small:

$$p(t, s; t', s') \sim e^{-\frac{d^2(s,s',t)}{2(t'-t)}} \frac{H(t, s; t', s')}{\sqrt{2\pi(t'-t)}s'\sigma(s', t')}$$

- $d(s, s', t) = \left| \int_s^{s'} \frac{d\xi}{\xi\sigma(\xi,t)} \right|$: geodesic distance between s to s'
- H: series expansion of heat kernel coefficients

Laplace type integral

Approximating the transition density using the heat kernel expansion, we end up with a Laplace type integral.

- Heat kernel expansion for product of transition densities

$$\prod_{i=1}^{n} p(t_{i-1}, s_{i-1}; t_i, s_i) \sim e^{-\frac{D_n(\mathbf{s},\mathbf{t})}{2\Delta t}} \prod_{i=1}^{n} \frac{H(t_{i-1}, s_{i-1}; t_i, s_i)}{\sqrt{2\pi\Delta t} s_i \sigma(s_i, t_i)},$$

where $D_n(\mathbf{s}, \mathbf{t}) = \sum_{i=1}^{n} d^2(s_{i-1}, s_i, t_{i-1})$.

- Laplace type integral

$$\mathbb{E}[(S_T - K)^+] \approx \int_{s_n \geq K} (s_n - K)\, e^{-\frac{D_n(\mathbf{s},\mathbf{t})}{2\Delta t}} \prod_{i=1}^{n} \frac{H(t_{i-1}, s_{i-1}; t_i, s_i)}{\sqrt{2\pi\Delta t} s_i \sigma(s_i, t_i)} d\mathbf{s}$$

Minimization problem for discrete-time MLP

As $\Delta t \to 0^+$, the main contribution of the Laplace type integral comes from the solution of the minimization problem:

$$\min_{\mathbf{s}} \frac{1}{2\Delta t} \sum_{i=1}^{n} d^2(s_{i-1}, s_i, t_{i-1})$$

subject to $s_n = K$.

Continuous-time limit

Since

$$d(s_{i-1}, s_i, t_{i-1}) \approx \left| \frac{\Delta s_i}{a(s_{i-1}, t_{i-1})} \right|,$$

we have

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{1}{2\Delta t} \sum_{i=1}^{n} d^2(s_{i-1}, s_i, t_{i-1}) &= \lim_{\Delta t \to 0} \frac{1}{2} \sum_{k=1}^{n} \left| \frac{\frac{\Delta s_i}{\Delta t}}{a(s_{i-1}, t_{i-1})} \right|^2 \Delta t \\ &= \frac{1}{2} \int_0^T \left| \frac{\dot{s}(t)}{a(s(t), t)} \right|^2 dt \end{aligned}$$

The variational most-likely-path

In the limit as Δt approaches zero, the minimization problem becomes the following variational problem

$$\min_{s(t)} \frac{1}{2} \int_0^T \left[\frac{\dot{s}(t)}{a(s(t),t)} \right]^2 dt \tag{9}$$

subject to

$$s(0) = s_0 \quad \text{and} \quad s(T) = K. \tag{10}$$

We call the solution to (9):(10), the *variational most-likely-path* (vMLP).

The Euler-Lagrange equation

The variational most-likely-path (vMLP) for a European option thus solves the Euler-Lagrange equation for the variational problem (9):(10) which can be written as

$$\frac{d}{dt}\left(\frac{\dot{s}}{a}\right) = \frac{\partial_t a}{a}\,\frac{\dot{s}}{a} \tag{11}$$

with initial and terminal conditions

$$s(0) = s_0, \quad s(T) = K.$$

Matching with Black-Scholes

Once the vMLP $s^*(t)$ is obtained, the call price is approximately

$$C(s_0, K, T) \sim e^{-\frac{1}{2}\int_0^T \left[\frac{\dot{s}^*(t)}{a(s^*(t),t)}\right]^2 dt}.$$

Recall the call price approximation under Black-Scholes:

$$C_{BS}(s, t, K, T, \sigma_{BS}) \approx \sqrt{s\,K}\,\frac{\tau^{\frac{3}{2}}}{\sqrt{2\pi}}\,e^{-\frac{k^2}{2\sigma_{BS}^2\tau}}\,\frac{\sigma_{BS}^3}{k^2}.$$

By matching exponents to leading order, we obtain

Implied volatility at lowest order

$$\frac{1}{\sigma_{BS,0}^2\, T} = \frac{1}{|k|}\int_0^T \left[\frac{\dot{s}^*(t)}{a(s^*(t), t)}\right]^2 dt \qquad (12)$$

Solution of the variational problem

With the change of variables

$$x(t) := \log s(t)/s_0; \quad a(s(t), t) = s(t)\, \sigma(x(t), t),$$

the Euler-Lagrange equation (11) becomes

$$\frac{d}{dt}\left\{\log\left(\frac{\dot{x}(t)}{\sigma(x(t), t)}\right)\right\} = \partial_t \log \sigma(x, t)|_{x=x(t)} =: f(x(t), t) \quad (13)$$

with boundary conditions $x(0) = 0$, $x(T) = k$.

- Note that for any given closed-form parameterization of the local volatility surface, $f(\cdot)$ is known in closed-form. $f(\cdot)$ is a measure of the time-inhomogeneity of the local volatility surface. In particular, if the local volatility function is time-independent, $f(\cdot) = 0$.

A recursive expression for the vMLP

Integrating (13) twice, we obtain

$$x(t) = k\, \frac{\int_0^t du\, \sigma(x(u), u)\, \exp\left\{\int_0^u f(x(s), s)\, ds\right\}}{\int_0^T du\, \sigma(x(u), u)\, \exp\left\{\int_0^u f(x(s), s)\, ds\right\}}. \tag{14}$$

- Equation (14) leads to an efficient fixed-point algorithm for finding the vMLP reminiscent of [Reghai].
- The natural choice of first guess is just the straight line $x_0(t) = k\, t/T$.
- Note that the vMLP for $k = 0$ ($K = s_0$) is $x(t) = 0$.

Implied volatility in terms of local volatility

In terms of the local volatility $\sigma(x,t)$, equation (12) may be rewritten as

Implied volatility in terms of local volatility

$$\sigma_{BS,0} = \frac{\frac{1}{T}\int_0^T \sigma(x(t),t)\,\exp\left\{\int_0^t f(x(s),s)\,ds\right\}\,dt}{\sqrt{\frac{1}{T}\int_0^T \exp\left\{2\int_0^t f(x(s),s)\,ds\right\}\,dt}} \tag{15}$$

Time-separable local volatility

Suppose $\sigma(x,t) = \sigma(x)\,\theta(t)$ for some functions $\sigma(\cdot)$ and $\theta(\cdot)$. Then by the definition of $f(\cdot)$,

$$f(x,t) = \frac{\theta'(t)}{\theta(t)}.$$

With $\phi(x) = \int_0^x \frac{d\xi}{\sigma(\xi)}$, the solution for the vMLP is

$$x(t) = \phi^{-1}\left(\phi(k)\,\frac{\int_0^t \theta^2(s)\,ds}{\int_0^T \theta^2(s)\,ds}\right). \tag{16}$$

Time-separable local volatility

Equation (15) then becomes

Implied volatility in a time-separable local volatility model

$$\sigma_{BS,0} = \frac{\frac{1}{T}\int_0^T \sigma(x(t))\,\theta(t)\,dt}{\sqrt{\frac{1}{T}\int_0^T \theta^2(t)\,dt}} \tag{17}$$

Time-homogeneous local volatility

- If $\sigma(x,t) = \sigma(x)$ is time-homogeneous, $\theta(\cdot)$ is constant, and

$$\sigma_{BS,0} = \frac{1}{T} \int_0^T \sigma(x(t))\, dt. \tag{18}$$

- At first sight, (18) seems to differ from the BBF formula.
  - In (18), implied volatility is expressed as an arithmetic mean and in the BBF formula as a harmonic mean.
- We show in [Gatheral and Wang] that the two expressions are in fact equivalent.

Time-changed time-homogenous local volatility

The time-separable case can be reduced to the time-homogeneous case by the simple deterministic time-change

$$\tau(t) = \int_0^t \theta^2(s)\, ds.$$

- We show that the vMLP (16) in the time-separable case is just the time-changed version of the vMLP in the time-homogeneous case.

Accuracy of our implied volatility approximation

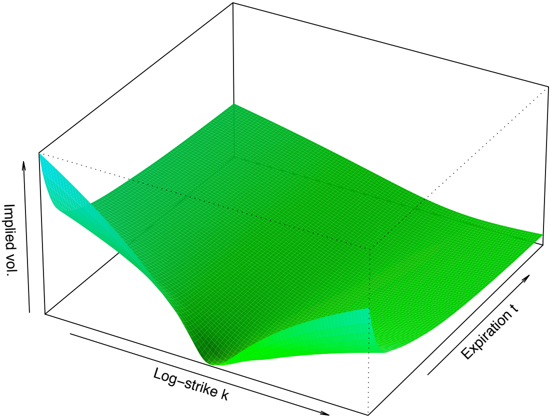
- Thus our approximation (15) behaves correctly under a deterministic time-change.
- It follows that the numerical accuracy of our approximation must be just as great in the time-separable case as it is in the time-homogeneous case.
- We know that the BBF formula is typically highly accurate in the time-homogeneous case.
- It follows that (15) is also typically highly accurate in the time-separable case.
    - No point in repeating the time-separable numerical examples in [GHLOW]!

Accuracy of our implied volatility approximation

- We have just established that our new approximation is highly accurate in the case of time-separable local volatility.
- This motivates us to test our new formula with a local volatility function that is both much more realistic and more difficult.
  - Difficult in the sense that its derivatives at $t = 0$ do not exist.
  - In particular, small-time expansions are not possible.

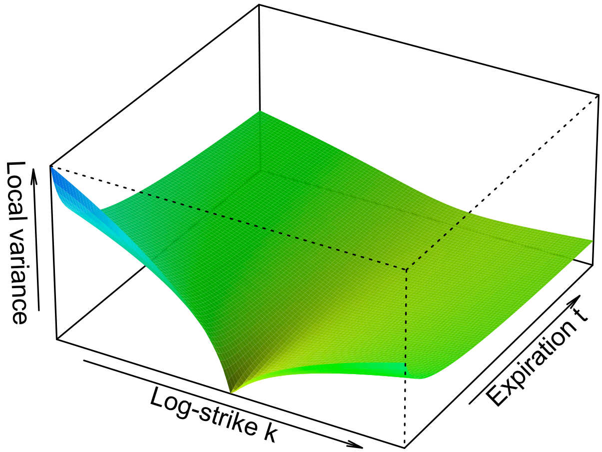
Figure 3.2: 3D plot of volatility surface

Here's a 3D plot of the volatility surface as of September 15, 2005:



$k := \log K/F$ is the log-strike and t is time to expiry.

3D plot of approximate local volatility surface



Local volatility surface parameterization

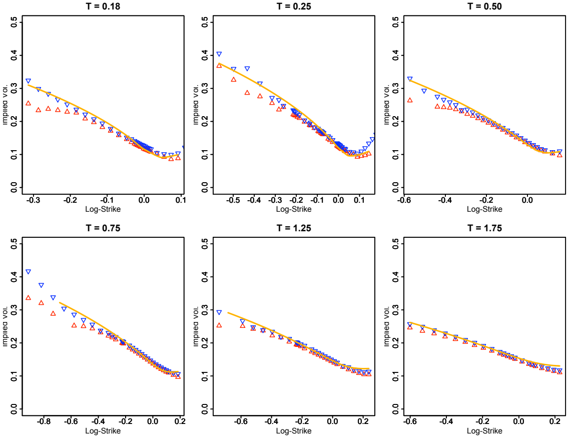
$$\sigma^2(k,t) = a + b\left\{\rho\left(\frac{k}{\sqrt{t}} - m\right) + \sqrt{\left(\frac{k}{\sqrt{t}} - m\right)^2 + \sigma^2\,t}\right\}$$

with

$$\begin{aligned} a &= 0.0012 \\ b &= 0.1634 \\ \sigma &= 0.1029 \\ \rho &= -0.5555 \\ m &= 0.0439 \end{aligned}$$

- Each slice is SVI.

Picture for the sceptical



Orange lines are from PDE computations, red and blue triangles are bid and offered vols respectively. Fits are not too bad!

Numerical tests

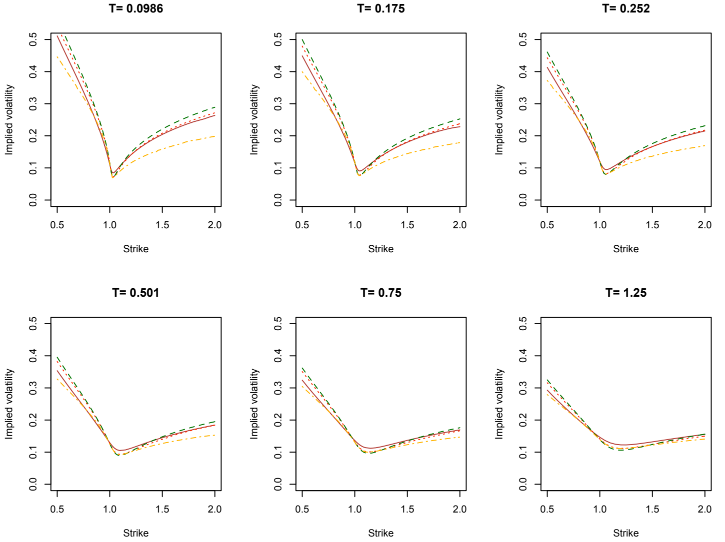
In the following slide, we compare various implied volatility approximations with an accurate estimate from numerical PDE:

- The brown solid lines are from PDE computations
- The red dotted lines are the vMLP estimates $\sigma_{BS,0}$
- The orange dash-dotted lines are from Adil Reghai's fixed point algorithm
- The green dashed lines correspond to a naïve extension (BBFe) of the BBF formula:

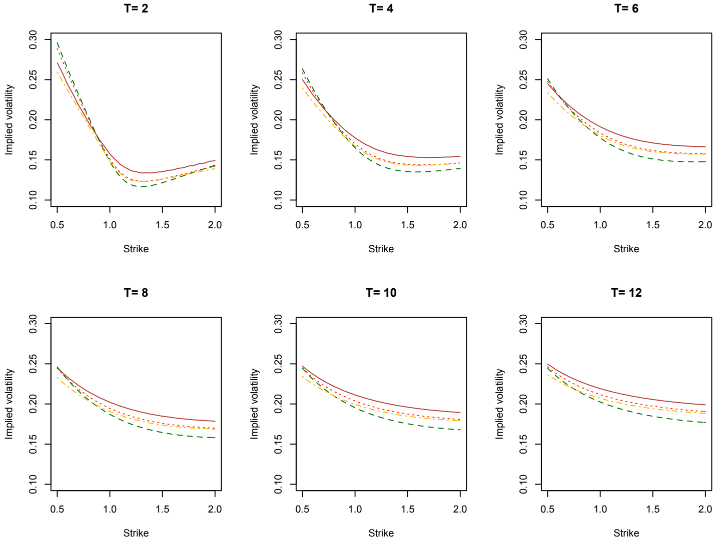
$$\frac{1}{\sigma_{BS}(k,T)} \approx \int_0^1 \frac{d\alpha}{\sigma_\ell(\alpha\, k, \alpha\, T)}$$

Expirations shown are actual SPX market expirations as of September 15, 2005.

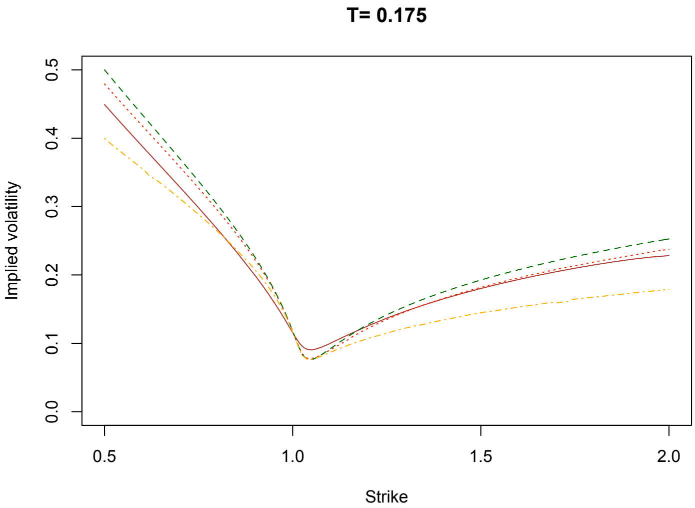
Comparison of approximations



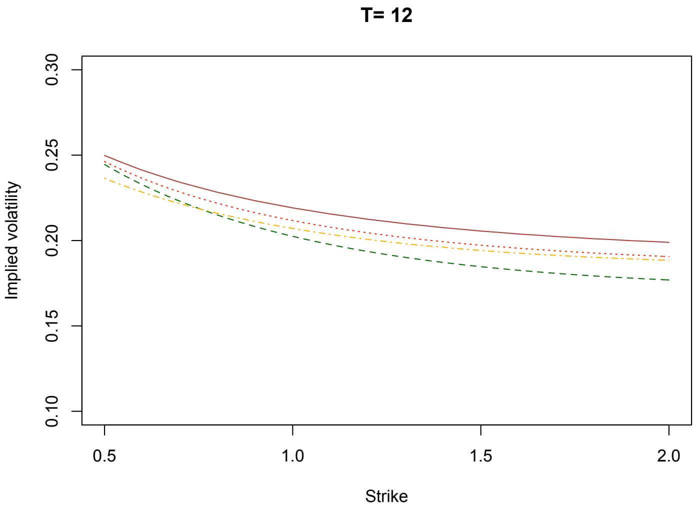
Longer-dated smiles



Zoomed view of short expiration smile



Zoomed view of long-dated smile



Observations

- Our vMLP approximation dominates the alternative most-likely-path approximations
- We note that BBFe does better than Reghai for shorter expirations and Reghai does better than BBFe for longer expirations
  - Variational MLP does better than either!

Convexity error

- For all three most-likely-path approximations, there is consistently a large approximation error at the cusp of the smile.
- As noted in [Guyon and Henry-Labordère], this is because the most-likely-path technique fails when there is substantial curvature in the local volatility function.
  - A convexity correction is required.
  - Fixing a time t, one cannot reasonably approximate an integration over all possible prices of the underlying x_t by one point, the most-likely point $\tilde{x}(t)$.
- The price to be paid for improved accuracy is in computational effort; a brute-force numerical PDE computation may be faster (and more accurate).
- In contrast, our fixed-point vMLP algorithm is very fast, typically converging within 3 or 4 iterations.

Summary and conclusion

- We have derived a new most-likely-path estimate which we call *variational MLP* for approximating the implied volatility surface given a local volatility function.
- The vMLP estimate is a natural extension of the BBF formula that behaves correctly under a deterministic time-change, in contrast to prior choices of most-likely-path.
- Our numerical tests indicate that the vMLP estimate outperforms two competing definitions of most-likely-path: a popular definition due to Adil Reghai and a naïve extension of the BBF formula.

- How to improve the accuracy of our variational MLP estimate by for example better approximating the integration over x_t is left for future research.

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