Market Impact with Autocorrelated Order Flow Under Perfect Competition: The Donier Model

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Terminology

- By *metaorder*, we mean an order that is sufficiently large that it cannot be filled immediately without eating into the order book.
    - Nowadays, this means just about any order.
    - Such orders need to be split.
- We refer to each component of a metaorder as a *child order*.
- By the *metaorder impact profile* (or just *impact profile*), we mean the average path of the stock price during and after execution of a metaorder.
- *Completion* refers to the timestamp of the last child order of a metaorder.

Schematic of the metaorder impact profile

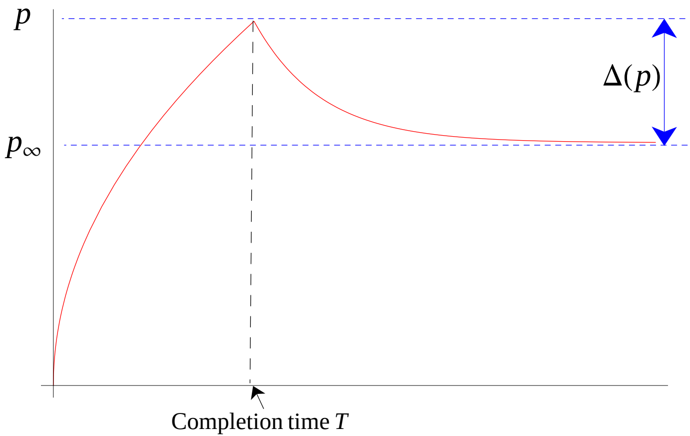


Figure 1: The metaorder impact profile

The impact profile

- When a buy metaorder is sent, its immediate effect is to move the price upwards (to p say).
- After completion, the price reverts to some price p_∞.
- Market impact then has two components, one transient and one permanent.
- Knowledge of the metaorder impact profile is key to the derivation of optimal execution strategies.

- The Donier model provides a framework for understanding and quantifying the impact profile.

Price impact of metaorders

It is now (fairly) well-accepted that:

- Metaorder price impact is a power-law function of the quantity, close to square-root.
- Prior to completion, the impact profile is a power-law function of time, close to square-root.
- After completion, market impact decays, possibly to a permanent level.

Impact of proprietary metaorders (from Tóth et al.)

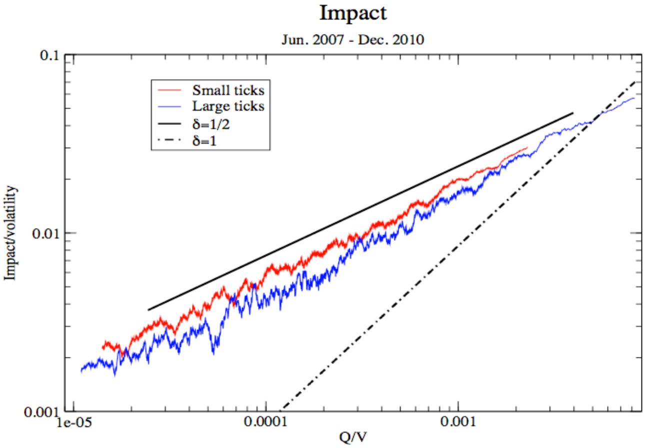


Figure 2: Log-log plot of the volatility-adjusted price impact vs the ratio Q/V

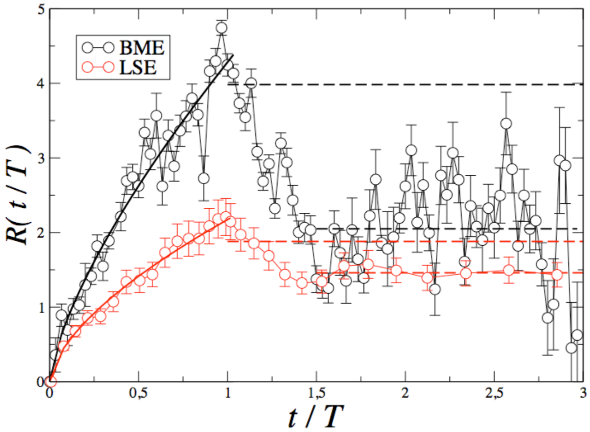
Notes on Figure 2

- In Figure 2 which is taken from [Tóth, Bouchaud et al.], we see the impact of metaorders for CFM[1] proprietary trades on futures markets, in the period June 2007 to December 2010.
  - Impact is measured as the average execution shortfall of a metaorder of size Q.
  - The sample studied contained nearly 500,000 trades.
- We see that the market impact is empirically power-law (roughly square-root) for metaorders with a range of sizes spanning two to three orders of magnitude!

[1] Capital Fund Management (CFM) is a well-known large Paris-based hedge fund.

Empirical impact profile (from Moro et al.)

Figure 3: Average path of the stock price during execution of a metaorder on two exchanges



Empirically observed impact profile

From Figure 3, we see that

- There is reversion of the stock price after completion of the metaorder.
- Some component of the market impact of the metaorder appears to be permanent.
- The path of the price prior to completion looks like a power law.
  - From [Moro et al.]

$$
\begin{aligned}
m_t - m_0 &\approx (4.28 \pm 0.21)\left(\frac{t}{T}\right)^{0.71\pm0.03} && \text{(BME)}\\
m_t - m_0 &\approx (2.13 \pm 0.05)\left(\frac{t}{T}\right)^{0.62\pm0.02} && \text{(LSE)}
\end{aligned}
$$

where T is the duration of the metaorder.

Summary of empirical observations

- The square-root formula gives an amazingly accurate rough estimate of the cost of executing an order.
- During execution of a metaorder, the price moves on average roughly according to $(t/T)^{2/3}$.
- Immediately after completion of a metaorder, the price begins to revert.
- The impact profile seems to exhibit scale invariance (which is implicit in Figure 3).

The Lillo, Mike, Farmer model of order splitting

- Let ϵ_t be the sign of the child order observed at time t.
- Then the autocorrelation function is given by $\rho(\tau) = \langle \epsilon_t \, \epsilon_{t+\tau} \rangle$.
  - By assumption, if two child orders come from different metaorders, their order signs are uncorrelated.
- $p(L)$ is the probability that a metaorder has length L.
- We assume there are always N active metaorders.

The power-law case

In the realistic case where metaorder sizes L are power-law distributed so

$$p(L) \sim L^{-(1+\gamma)}$$

we find

$$\rho(\tau) \sim N^{\gamma-2}\,\tau^{1-\gamma}.$$

In particular, if $\gamma = 3/2$ as is more or less the case empirically for many stocks, we have

$$\rho(\tau) \sim \frac{1}{\sqrt{\tau}}.$$

- The LMF model gives a link between the distribution of order sizes and the autocorrelation function of order signs.

Empirical confirmation

- [Tóth, Lillo et al.] perform a careful analysis of order flow data from the London Stock Exchange containing exchange membership identifiers.
- They conclude that order splitting is indeed the dominant cause of the long memory of the order sign process.

The Glosten and Milgrom sequential trade model

- In the [Glosten and Milgrom] model, the market maker $\mathcal{M}$ learns the informed trader $\mathcal{I}$'s information by observing the order flow.
  - If there are more buys than sells over time, $\mathcal{M}$ sets the price higher.
- Under *perfect competition*:
  - $\mathcal{M}$ posts bid and ask prices $B = \mathbb{E}[V|Sell]$ and $A = \mathbb{E}[V|Buy]$ where V denotes the efficient price.
  - The spread $s = A - B$ is proportional to the probability μ of informed trading.

Dynamical properties of the Glosten and Milgrom model

- The trade price series is a martingale.
  - Both bid and ask prices are expectations conditioned on an expanding information set (the time series of trade signs):

$$\begin{aligned} B_k &= \mathbb{E}[V|\mathcal{F}_k, \epsilon_k = -1] \\ A_k &= \mathbb{E}[V|\mathcal{F}_k, \epsilon_k = +1] \end{aligned}$$

- Order signs are predictable: $\mathbb{E}[\epsilon_k|\mathcal{F}_k] \neq 0$ in general.
- Orders are serially correlated because informed traders always trade in the same direction.
- There is market impact in this model. A buy causes both the bid and the offer to increase.

The FGLW market impact model

In the model of [Farmer, Gerig, Lillo, and Waelbroeck] (FGLW from now on), there is a market maker $\mathcal{M}$, informed traders $\mathcal{I}$ and uninformed (or noise) traders $\mathcal{U}$. Informed traders trade using metaorders.

The authors show that the typical impact profile associated with the execution of a metaorder may be recovered starting from two assumptions:

- **The Martingale Condition**: The price process is a martingale.
  - $\mathcal{M}$ does not know how long a given metaorder will continue.
- **The Fair Pricing Condition**: On average, the price reverts after completion of the metaorder to a level equal to the average price paid by $\mathcal{I}$.
  - If metaorder sizes are power-law distributed with exponent γ, the price reverts on average to a level which is a factor $1/\gamma$ of the peak price reached at completion.

Empirical confirmation of FGLW

- In a (very!) recent paper, [Bershova and Rakhlin] perform an empirical study of a proprietary dataset of large institutional equity orders.
- They broadly confirm the predictions of FGLW, including the power-law impact profile, the reversion level of 2/3 and roughly square-root permanent impact.
- They also study reversion of market impact in detail.

The CFM model of latent supply and demand

- [Tóth, Bouchaud et al.] introduce the concept of *latent* supply and demand, to be distinguished from the visible supply/demand profile associated with the limit order book.
- [Tóth, Bouchaud et al.] note that if latent supply and demand is roughly linear in price over some reasonable range of prices, market impact should be roughly square-root.

The Donier model

In the model of [Donier]:

- As before, the market consists of informed traders $\mathcal{I}$, noise traders $\mathcal{U}$, and market makers $\mathcal{M}$.
- There is *perfect competition* between market makers. We may thus suppose there is only one market maker $\mathcal{M}$.
- Informed traders submit metaorders generating autocorrelation in the order sign process according to the order-splitting model of [Lillo, Mike and Farmer].
- $\mathcal{M}$ has *perfect information* and can distinguish informed and uniformed child orders as in the colored print model of [Gerig, Farmer and Lillo].
    - As before, $\mathcal{M}$ does not know in advance when $\mathcal{I}$'s metaorder will complete.
    - $\mathcal{M}$ does however know the distribution of metaorder sizes.

Market making strategy

The market maker reacts to a child order as follows:

- If the child order is uniformed, $\mathcal{M}$ responds by refilling the order book so as to restore it to its original state.
- If the child order is informed, $\mathcal{M}$ lets the child order eat into the order book.
    - The zero profit condition imposes the quantity v_p that should be available at the price level p.
- Only informed trades can cause the price to move.

Computation of v_p

- Let ℓ denote the (unknown) length of the metaorder.
- $\mathcal{M}$ posts limit orders such that the quantity $v_p = L_p - L_{p-1}$ available at price p satisfies

$$p = \mathbb{E}[p_\infty \,|\ell \geq L_p] \tag{1}$$

where $p_\infty = \lim_{t\to\infty} p_t$.

  - Note that the v_p limit orders could be posted either in advance or at the time in response to incoming market orders. The v_p thus represents *latent* limit order supply.
  - Strictly speaking, our argument works only if informed orders have length L_k for some k. We believe that a more careful analysis will not substantially change the results.

Imperfect information

- We have assumed perfect information for convenience; only informed trades move the price which is thus a deterministic function of L.
  - The impact profile and in particular the final price p and the reversion price p_∞ are deterministic.
- In reality, $\mathcal{M}$ cannot tell whether a given child order is informed or not. Information is asymmetric and noise trades also move the price.
  - The deterministic impact profile, p and p_∞ should then be viewed as expectations of actual prices.
- $\mathcal{I}$ is informed only in the sense that he knows what the size of his own metaorder is. This is of course $\mathcal{I}$'s private information.

- Let ℓ be the (*a priori* unknown) length of the metaorder and p_{max} the price reached at completion.
- Then, either
  - $\ell > L + v_p$ and the metaorder exhausts the available quantity at p, causing the price to increase to $p + 1$, or
  - $L \leq \ell \leq L + v_p$, $p = p_{\max}$, the metaorder completes, and the price reverts to p_∞.
- Denote the probability that the metaorder continues by q.
- Then because p_t is a martingale,

$$p = (1 - q)\, p_\infty + q\, (p + 1) \tag{2}$$

and the decay after completion (*transient impact*) is

$$\Delta(p) := p - p_\infty = \frac{q}{1 - q}. \tag{3}$$

- Equation (2) is essentially the Martingale Condition of FGLW.

Relation between order size distribution and p_∞

- Denote the (informed) order size required to consume limit orders up to price p by $L_p = \sum_{k=0}^{p} v_p$ and the tail distribution function of metaorder sizes by $\tilde{F}(L) = \mathbb{P}(\ell \geq L)$.
- Then the continuation probability q is given by

$$q = \mathbb{P}(\ell \geq L_p | \ell \geq L_{p-1}) = \frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1})}.$$

- Thus

$$\Delta(p) = \frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1}) - \tilde{F}(L_p)}. \tag{4}$$

Remarks

- Equation (4) relates the magnitude of transient impact to the distribution of metaorder sizes.
- (4) is valid for any given metaorder size distribution, not just for the power-law case explored in the following.
- Note that the price must revert immediately to p_∞ after completion if $\mathcal{M}$ can tell when the metaorder ends.
  - In real markets, such information is partial and is inferred from observations; the price after completion decays over time to p_∞.

- Suppose further that $\mathcal{M}$ sets his prices so as to target zero expected profit for any order size. Then

$$L_p \, p_\infty = \sum_{k=0}^{p} k \, v_k \tag{5}$$

  - This condition is not strictly imposed by the perfect competition assumption but can be obtained by supposing for example that $\mathcal{M}$ is risk averse and minimizes his P&L volatility.
- Equation (5) is the Fair Pricing condition of FGLW.
  - However (5) follows from the assumption of perfect competition between market makers. There is no notion of fair pricing here.

Price reversion

- The Fair Pricing Condition (5) gives

$$\Delta(p) = p - p_\infty = p - \frac{1}{L_p}\sum_{k=0}^{p} k\, v_k \tag{6}$$

which may be rewritten as

$$\Delta(p) = \frac{1}{L_p}\sum_{k=0}^{p-1} L_k. \tag{7}$$

- In particular, since $0 < L_k < L_p$ for all $k < p$, we must have $0 < \Delta(p) < p$.
  - That is, the price always reverts after completion, no matter what the distribution of metaorder sizes is.

Rôle of the bid-ask spread

- Equation (5) applied to the case $p = 0$ gives $p_\infty = 0$.
- The martingale condition then implies, as in FGLW, that the price cannot move.
  - This problem may be resolved by introducing a bid-ask spread.
- From (2), when the spread is a half-tick, zero expected profit per share imposes that

$$\frac{1}{2} = (1 - q)\, p_\infty + q\,(p + 1) = q$$

when $p = p_\infty = 0$. This then gives us the condition that

$$\mathbb{P}(\ell \geq L_0) = \frac{1}{2}$$

which fixes L_0 in terms of the spread.

  - In FGLW, L_0 is an undetermined parameter which is used to set the scale of market impact. In contrast, the perfect competition assumption imposes a connection between L_0 and the spread.

The latent order book

- The volume v_p that the market maker posts at price p can be interpreted as the latent volume that would emerge were the price to reach p.
- In the Donier model, the shape of the latent order book reflects the adaptive reaction of the market-maker under perfect competition to the distribution of metaorder sizes, as estimated for example from the order flow autocorrelation function.
    - In particular, according to the LMF order-splitting model, if $\mathbb{P}(\ell > L) \sim L^{-\gamma}$, the autocorrelation function of order flow decays as

$$\rho(\tau) \sim \frac{1}{\tau^{\gamma-1}}.$$

Recursion for L_p

Equation (4) reads

$$\Delta(p) = \frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1}) - \tilde{F}(L_p)}$$

and equation (7) reads

$$\Delta(p) = \frac{1}{L_p} \sum_{k=0}^{p-1} L_k.$$

Equating these two gives

$$\frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1}) - \tilde{F}(L_p)} = \frac{1}{L_p} \sum_{k=0}^{p-1} L_k. \tag{8}$$

from which the L_p may be computed recursively starting with (*e.g.*) $\tilde{F}(L_0) = 1/2$.

Solving for the metaorder impact profile

- The latent order book is then given by

$$v_p = L_p - L_{p-1}.$$

- Plotting p vs L_p gives the impact profile prior to completion.
- The reversion level is given by

$$\Delta(p) = \frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1}) - \tilde{F}(L_p)}$$

where p is the maximum price reached.

- We can generate the impact profile for any choice of $\tilde{F}(\cdot)$.

Example: The zeta distribution

Suppose that the probability of an order of length ℓ is given by

$$f(\ell) = \frac{\ell^{-\gamma+1}}{\zeta(\gamma+1)}$$

Then

$$\tilde{F}(\ell) = \sum_{j=\ell}^{\infty} f(j) = \frac{\zeta(\gamma+1,\ell)}{\zeta(\gamma+1)} \sim \ell^{-\gamma} \text{ as } \ell \to \infty.$$

Set $\gamma = 3/2$. The condition $\tilde{F}(L_0) = 1/2$ gives

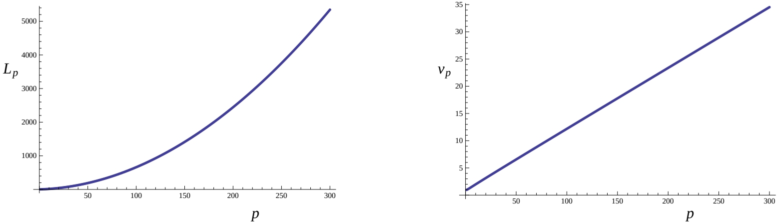
$$L_0 = 1.40541.$$

We then find recursively

$$\{L_1, L_2, L_3, \dots\} = \{2.37, 3.43, 4.59, \dots\}. \tag{9}$$

Graphs of L_p and v_p

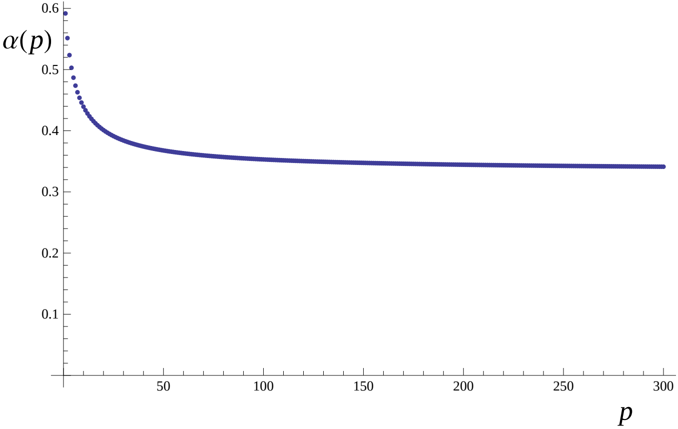
Figure 4: Plots of the cumulative latent order depth L_p (left) and the latent order density v_p (right) vs p



- L_p is roughly quadratic and $v_p = \Delta L_p$ is roughly linear.

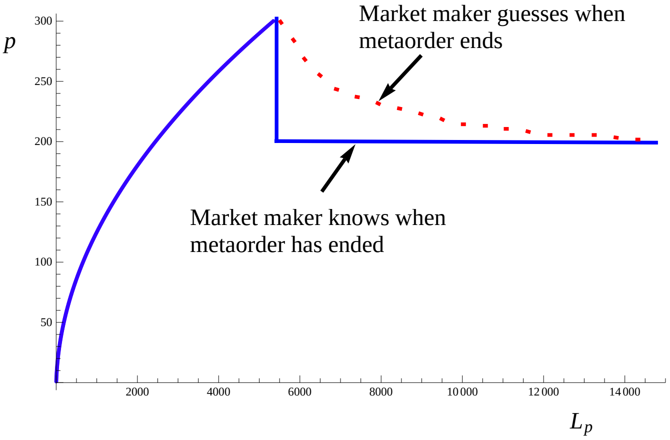
The reversion level

Figure 5: Graph of the relative reversion $\alpha(p) := \Delta(p)/p$ vs p



The impact profile: Power-law case

Figure 6: The metaorder impact profile for a trade of length L_{300}



Price reversion after completion

- The deterministic version of the Donier model gives us the impact profile prior to completion.
  - If $\mathcal{M}$ can tell when the metaorder has ended, the decay to p_∞ is instant.
  - Otherwise, we need to consider the information set available to $\mathcal{M}$. We will return to this later.

Asymptotic analysis: Power-law distribution

Rewrite (4) as

$$\frac{\tilde{F}(L_p)}{\tilde{F}(L_{p-1})} = \frac{1}{1 + \frac{1}{\Delta(p)}}.$$

Assume $\alpha(p) := \Delta(p)/p \to \alpha_\infty \in [0,1]$ as $p \to \infty$. Taking logs gives

$$\log \tilde{F}(L_p) - \log \tilde{F}(L_{p-1}) = -\log\left(1 + \frac{1}{\alpha_\infty p}\right)$$

Then

$$\log \tilde{F}(L_p) = -\sum_{k=1}^{p} \log\left(1 + \frac{1}{\alpha_\infty p}\right) \sim -\frac{1}{\alpha} \log p \text{ as } p \to \infty \quad (10)$$

The power-law case: Impact profile

If $\tilde{F}(L) \sim L^{-\gamma}$, (10) gives

$$p(L) \sim L^{\alpha_\infty \gamma} \text{ as } L \to \infty$$

- This gives the impact profile prior to completion (asymptotically for large L).

The Fair Pricing condition (5) can be approximated for large p as

$$p_\infty = \frac{1}{L_p} \int_0^{L_p} p(L)\, dL = \frac{p}{\alpha_\infty \gamma + 1}$$

Also, $p - \Delta(p) = (1 - \alpha_\infty)\, p$ so

$$\alpha_\infty = 1 - \frac{1}{\gamma} \quad \text{and} \quad p_\infty = \frac{1}{\gamma}\, p.$$

The power-law case: Latent order book

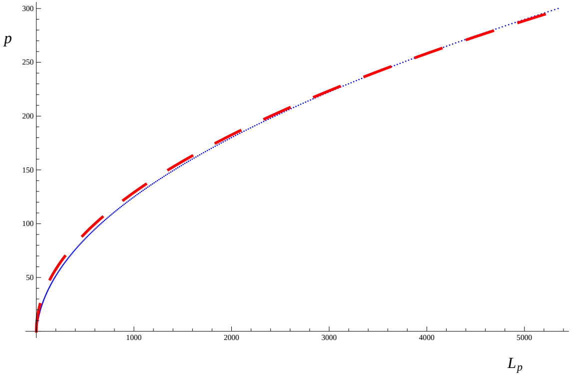
Since $p \sim L^{\gamma-1}$, we have

$$\begin{aligned} v_p = L_p - L_{p-1} &\sim p^{\frac{1}{\gamma-1}} - (p-1)^{\frac{1}{\gamma-1}} \\ &\sim \frac{1}{\gamma-1}\, p^{\frac{1}{\gamma-1}-1} \end{aligned}$$

- With $\gamma \approx 3/2$ consistent with one of the stylized facts, we obtain both
  - the linear latent order book of [Tóth, Bouchaud et al.], and
  - the metaorder impact profile of FGLW, in particular the reversion level $p_\infty = \frac{2}{3}\, p$.

Asymptotic profile vs exact profile

Figure 7: The blue points are the exact solution (9); the red dashed line is the asymptotic solution (10)



- The asymptotic solution is qualitatively very close to the exact solution.

Asymptotic analysis: Exponential distribution

Again assume $\alpha(p) := \Delta(p)/p \to \alpha_\infty \in [0,1]$ as $p \to \infty$. Then if $\tilde{F}(L) \sim e^{-\lambda L}$, (10) gives

$$-\lambda L_p \sim \log \tilde{F}(L_p) = -\sum_{k=1}^{p} \log\left(1 + \frac{1}{\alpha_\infty p}\right) \sim -\frac{1}{\alpha} \log p \text{ as } p \to \infty$$

Equation (7) then reads

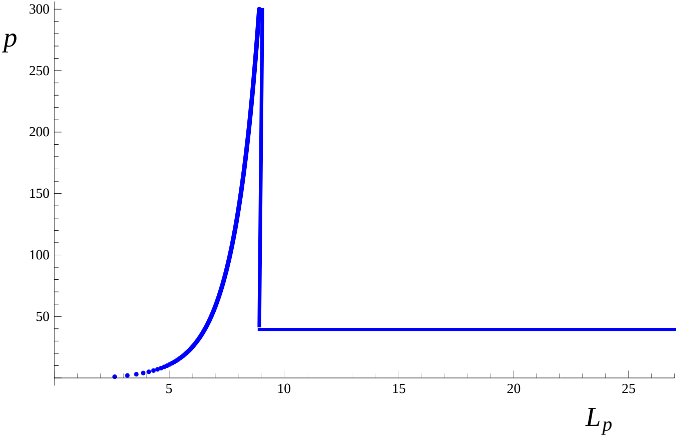
$$\alpha_\infty \sim \frac{1}{p} \frac{1}{L_p} \sum_{k=0}^{p-1} L_k = \frac{1}{p \log p} \sum_{k=0}^{p-1} \log k \to 1 \text{ as } p \to \infty$$

so the price reverts asymptotically to zero.

- This impact profile is completely inconsistent with empirical observation.
- We now confirm this with a numerical computation.

The impact profile: Exponential case

Figure 8: The metaorder impact profile for a trade of length L_{300}



Comparison with empirical impact profile

- Obviously the power-law profile is much more realistic.
  - The convex impact in Figure 8 is completely inconsistent with the empirical impact profile of for example [Moro et al.] (Figure 3).

Price reversion

- As indicated earlier, if $\mathcal{M}$ does not know if the order has ended, the price cannot suddenly revert to p_∞.
- However $\mathcal{M}$ knows both the participation rate π of $\mathcal{I}$ and the number n of informed child orders so far.
- Denote the event that the metaorder is still active by $\mathcal{A}$.
- Let m be the number of uniformed child orders since the last informed order and assume that orders arrive as Poisson processes.

- $\mathcal{M}$ may then use a Bayesian argument to compute $\mathbb{P}(\mathcal{A}|m)$ (it being implicit that we assume n informed orders already observed).

A Bayesian argument

We have

$$\mathbb{P}(m|\mathcal{A}) = (1-\pi)^m \text{ and } \mathbb{P}(m|\bar{\mathcal{A}}) = 1.$$

Also, note that

$$\mathbb{P}(\mathcal{A}) = \frac{\tilde{F}(n+1)}{\tilde{F}(n)}$$

is the probability that there are more informed child order to come, we have

$$\begin{aligned}
\mathbb{P}(m) &= \mathbb{P}(m|\mathcal{A})\,\mathbb{P}(\mathcal{A}) + \mathbb{P}(m|\bar{\mathcal{A}})\,\mathbb{P}(\bar{\mathcal{A}}) \\
&= (1-\pi)^m\,\mathbb{P}(\mathcal{A}) + (1-\mathbb{P}(\mathcal{A})) \\
&= 1 - \mathbb{P}(\mathcal{A})\left[1-(1-\pi)^m\right].
\end{aligned}$$

Then

$$\begin{aligned}\mathbb{P}(\mathcal{A}|m) &= \frac{\mathbb{P}(m|\mathcal{A})\,\mathbb{P}(\mathcal{A})}{\mathbb{P}(m)}\\ &= \frac{(1-\pi)^m\,\mathbb{P}(\mathcal{A})}{1-\mathbb{P}(\mathcal{A})\,[1-(1-\pi)^m]}.\end{aligned}$$

Let p_m denote the price after m uninformed child orders. *Perfect competition* imposes that

$$p_m = \mathbb{P}(\mathcal{A}|m)\,p_{\mathcal{A}} + \mathbb{P}(\bar{\mathcal{A}}|m)\,p_\infty$$

where $p_{\mathcal{A}}$ is the price if the metaorder continues. In particular,

$$p = p_0 = \mathbb{P}(\mathcal{A})\,p_{\mathcal{A}} + \mathbb{P}(\bar{\mathcal{A}})\,p_\infty = p_\infty + \mathbb{P}(\mathcal{A})\,(p_{\mathcal{A}} - p_\infty).$$

Then

$$\begin{aligned} p_m &= p_\infty + \mathbb{P}(\mathcal{A}|m)\,(p_\mathcal{A} - p_\infty) \\ &= p_\infty + \frac{\mathbb{P}(\mathcal{A}|m)}{\mathbb{P}(\mathcal{A})}\,(p - p_\infty) \\ &= p_\infty + \frac{(1-\pi)^m}{1 - \mathbb{P}(\mathcal{A})\,[1-(1-\pi)^m]}\,(p - p_\infty) \end{aligned} \tag{11}$$

- $p_0 = p$. The price is fair if no uninformed child orders are detected.
- $p_m \to p_\infty$ as $m \to \infty$. If no new informed child orders are ever detected, the fair price is p_∞.
- Decay from p to p_∞ is exponential but scale invariant:

$$p_m - p_\infty \sim e^{-m/\bar{m}}\,(p - p_\infty) \text{ with } \bar{m} = 1/\pi.$$

Power-law case

If $\tilde{F}(L) \sim L^{-\gamma}$, then as $n \to \infty$,

$$\mathbb{P}(\mathcal{A}) = \frac{\tilde{F}(n+1)}{\tilde{F}(n)} \sim \left(1 + \frac{1}{n}\right)^{-\gamma} \sim 1 - \frac{\gamma}{n}.$$

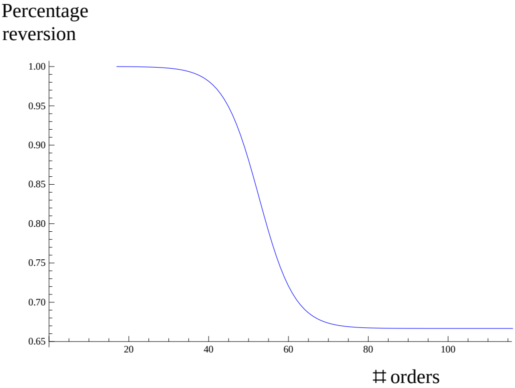
Then

$$\begin{aligned} p_m &= p_\infty + \frac{(1-\pi)^m}{1 - \mathbb{P}(\mathcal{A})\,[1-(1-\pi)^m]}\,(p - p_\infty) \\ &\sim p_\infty + \frac{(1-\pi)^m}{1 - \left(1 - \frac{\gamma}{n}\right)\,[1-(1-\pi)^m]}\,(p - p_\infty). \end{aligned}$$

We plot this price reversion profile in Figure 9.

IBM example impact profile

Figure 9: Typical price reversion profile



- $\mathcal{M}$ maintains the price at p until he is certain that the metaorder is no longer active. Then the price drops quickly to p_∞.

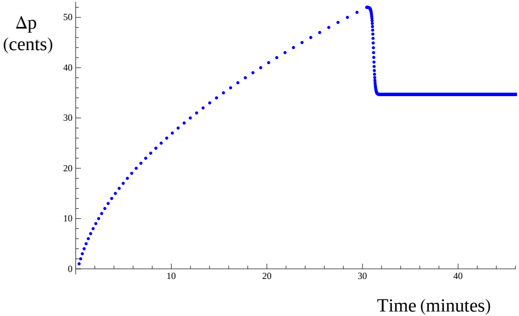
Numerical example: IBM

- Consider a trade of 1% of daily volume of IBM which is around $40,000$ shares, executed as a VWAP over 30 minutes ($\approx 1/13$ days).
- Participation rate $\pi \approx 13\%$.
- Assume power-law distributed order sizes with $\gamma = 3/2$.
- Further assume that each child order is for 200 shares (roughly the average trade size). Then there will be 200 child orders in this metaorder.
- The tick size is \$0.01 on a price of \$200.

- With these parameters, this stylized version of the Donier model predicts a peak price move of \$0.52 with reversion to \$0.35.
  - Assuming daily volatility of 2%, the square-root formula estimates \$0.40 so the numbers are of the right order of magnitude.

IBM example impact profile

Figure 10: The impact profile for 1% of DV traded over 30 minutes



- Scale invariance means that the impact profile for the same quantity traded over a different time interval (such as 2 hours) is obtained by dilation of the time axis.

Price manipulation

- For each new metaorder then, the price reverts exponentially to p_∞ after some characteristic time that is proportional to the completion time of the order.
  - The decay constant should be related to the precision of the estimator used by $\mathcal{M}$ to detect the end of the metaorder.
- In the power-law case, $p_\infty = \alpha_\infty\, p \sim L^{\gamma-1}$, so if $\gamma < 2$, there should be price manipulation in the sense of Huberman and Stanzl (see [Gatheral]).
  - An obvious strategy would be to buy $2\,N$ shares using metaorders of length N separated by an interval long enough to allow for full decay of the price to the permanent level. Then sell back using a metaorder of length $2\,N$.

Bare and renormalized impact profiles

- The impact profile we have drawn is the *bare* impact profile assuming that:
  - There are no other active metaorders,
  - $\mathcal{M}$ has no memory of previous metaorders.
- To be consistent with empirical studies such as that of [Moro et al.], we need the *renormalized* impact profile which corresponds to averaging unconditionally over a dataset of metaorders, ignoring the initial state of the market.

The renormalized impact profile

- In general, if a given trader A submits a child order, that order will either be in the same direction or the opposite direction to the net of other active metaorders (denoted by B).
- Denote the bare impact function by $I(\cdot)$ and the renormalized impact function by $\bar{I}(\cdot)$. Then assuming that B has already traded L shares, the impact of a single A child order is given by

$$\begin{aligned}\Delta\bar{I} &= I(L+1) - I(L) \approx I'(L) \text{ (same sign)}\\ \Delta\bar{I} &= -\{I(L-1) - I(L)\} \approx I'(L) \text{ (opposite sign).}\end{aligned}$$

- In general, the sign of B's metaorders will change several times during the execution of A's metaorder.

Let ℓ be the size of A's metaorder and assume that the participation rates of A and B are equal. Then

$$\bar{I}(\ell) \approx \sum_{i=1}^{N_\ell} L_i \, I'(L_i)$$

assuming that B trades $\sum_i L_i$ shares during A's metaorder execution (of ℓ shares) and B's order changes sign N_ℓ times.

Two limiting cases

Denote the typical size of B's orders by $\bar{L}$. Then

- If $\ell \ll \bar{L}$, $N_\ell = 1$ and
$$\bar{I}(\ell) \approx \ell \, I'(\ell) \, .$$
- If $\ell \gg \bar{L}$, $N_\ell = \ell / \bar{L}$ and
$$\bar{I}(\ell) \approx N_\ell \, \bar{L} \, I'\left(\bar{L}\right) = \ell \, I'\left(\bar{L}\right) \propto \ell.$$

Two limiting cases: Power-law impact

In the power-law case $I(L) = C\,L^\delta$ with $\delta = \gamma - 1 \approx \frac{1}{2}$, we have

$$\bar{I}(\ell) \approx \begin{cases} \delta\, I(\ell) & \text{if } \ell \ll \bar{L} \\ \delta\, \left(\frac{\bar{L}}{\ell}\right)^\delta\, I(\ell) & \text{if } \ell \gg \bar{L}. \end{cases}$$

- We see that renormalized market impact is always less than bare market impact.
  - However, permanent impact is always nonzero.
- In the limit $\ell \gg \bar{L}$, market impact $\bar{I}(\ell)$ is linear in ℓ.
  - So, for metaorders executed over long timescales, price manipulation is not possible.
  - The ratio of p_∞ / p is preserved ($2/3$ if $\delta = 1/2$).

Summary

- The Donier model framework has the following nice properties:
  - Both the Martingale Condition and (an approximate version of) the Fair Pricing Condition of FGLW follow from the assumption of perfect competition between market makers.
  - It provides a natural interpretation of the latent order book of [Tóth, Bouchaud et al.].
- We computed the difference between the bare and renormalized market impact functions.
- Price manipulation is not possible for long trades. It is possible that price manipulation is not possible in general.
- Further analysis is both possible and required ...

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