Optimal order execution

Jim Gatheral
(including joint work with Alexander Schied and Alla Slynko)

Petit Déjeuner de la Finance
April 15, 2010

References

[1] Robert Almgren
Equity market impact
Risk (2005) July, pp. 57-62

[2] Robert Almgren and Neil Chriss
Optimal execution of portfolio transactions
Journal of Risk (2001) Volume 3 pp. 5–40

[3] Aurélien Alfonsi, Antje Fruth and Alexander Schied
Optimal execution strategies in limit order books with general shape functions
Quantitative Finance (2010) Volume 10, Issue 2 pp. 143–157

[4] Aurélien Alfonsi, Alexander Schied and Alla Slynko
Order book resilience, price manipulation, and the positive portfolio problem
http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1498514 (2009)

[5] P.A. Forsyth, J.S. Kennedy, S. T. Tse, and H. Windcliff
Optimal Trade Execution: A Mean - Quadratic-Variation Approach
University of Waterloo (2009)

References

[6] Jim Gatheral
No-dynamic-arbitrage and market impact .
Forthcoming *Quantitative Finance.*

[7] Jim Gatheral, Alexander Schied and Alla Slynko
Transient linear price impact and Fredholm integral equations
http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1531466

[8] Anna Obizhaeva and Jiang Wang
Optimal trading strategy and supply/demand dynamics.
MIT working paper (2005)

[9] Silviu Predoiu, Gennady Shaikhet and Steven Shreve
Optimal execution in a general one-sided limit-order book
http://papers.ssrn.com

[10] Alexander Weiss
Executing large orders in a microscopic market model
http://arxiv.org/abs/0904.4131v2

Overview of this talk

- Statement of the optimal execution problem
- Statically and dynamically optimal strategies
  - Variational calculus
  - The dynamic programming principle and HJB
- The Almgren-Chriss framework and 2001 model
- Almgren's 2005 model
- The Obizhaeva and Wang model
- The Alfonsi and Schied model
- Price manipulation and existence of optimal strategies
- Transient linear price impact

Statement of the problem

- Given a model for the evolution of the stock price, we would like to find an optimal strategy for trading stock, the strategy that minimizes some cost function over all permissible strategies.
  - We will specialize to the case of stock liquidation where the initial position $x_0 = X$ and the final position $x_T = 0$.
- A *static* strategy is one determined in advance of trading.
- A *dynamic* strategy is one that depends on the state of the market during execution of the order, *i.e.* on the stock price.
  - Delta-hedging is an example of a dynamic strategy. VWAP is an example of a static strategy.
- It will turn out, surprisingly, that in many models, a statically optimal strategy is also dynamically optimal.

The Euler-Lagrange equation

Suppose that the strategy x_t minimizes the cost functional

$$\mathcal{C}[x] = \int_0^T L(t, x_t, \dot{x}_t)\, dt$$

with boundary conditions $x_0 = X$, $x_T = 0$.

Then we have the Euler-Lagrange equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

Bellman's principle of optimality

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

(See Bellman, 1957, Chap. III.3.)

Stochastic optimal control

Consider a cost functional of the form

$$J_t = \mathbb{E}\left[\int_t^T h(t, y_s, v_s)\, ds\right]$$

where y_s is a state vector, v_s is a vector-valued control and the evolution of the system is determined by a stochastic differential equation (SDE):

$$dy_t = f(t, y_t, v_t)\, dt + \sigma(t, y_t, v_t)\, dZ_t$$

Then J satisfies the HJB equation

$$\frac{\partial J}{\partial t} + \min_{v \in \mathcal{G}} \{\mathcal{L}_t^v J + h(t, y_t, v)\} = 0$$

where $\mathcal{L}_t^v$ is the infinitesimal generator of the Itô diffusion:

$$\mathcal{L}_t^v = \frac{1}{2}\sigma(t, y, v)^2\, \partial_{y,y} + f(t, y, v)\, \partial_y$$

Deterministic and stochastic optimal control

- In *deterministic* optimal control, the evolution of the state vector is deterministic.
- In *stochastic* optimal control, the evolution of the state vector is stochastic.

Almgren and Chriss

Almgren and Chriss [**?**] model market impact and slippage as follows. The stock price S_t evolves as

$$dS_t = \sigma \, dZ_t$$

and the price $\tilde{S}_t$ at which we transact is given by

$$\tilde{S}_t = S_t + \eta \, v_t$$

where $v_t := -\dot{x}_t$ is the rate of trading.

The state vector is $y_t = \{S_t, x_t\}$. The components of the state vector evolve as

$$dS_t = \sigma \, dZ_t; \; dx_t = -v_t \, dt;$$

Cost of trading

The risk-unadjusted cost of trading (with no penalty for risk) is given by

$$\mathcal{C}_t = \mathbb{E}_t \left[\int_t^T \tilde{S}_s \, v_s \, ds \right] = \mathbb{E}_t \left[\int_t^T (S_s + \eta \, v_s) \, v_s \, ds \right]$$

The HJB equation becomes

$$\frac{\partial \mathcal{C}}{\partial t} + \frac{1}{2} \sigma^2 \, \mathcal{C}_{S,S} + \min_{v \in \mathcal{G}} \{ -\mathcal{C}_x \, v_t + (S_t + \eta \, v_t) \, v_t \} = 0 \qquad (1)$$

and the optimal choice of v_t (the *first order condition*) is

$$v_t^* = \frac{1}{2 \, \eta} \, (\mathcal{C}_x - S_t)$$

Substituting back into (**??**) and defining $\tilde{\mathcal{C}} := \mathcal{C} - x\,S$ gives the equation for the cost function:

$$\left(\tilde{\mathcal{C}}_x\right)^2 = 4\,\eta\,\tilde{\mathcal{C}}_t \tag{2}$$

with boundary conditions $\tilde{\mathcal{C}}(T, y_T) = \tilde{\mathcal{C}}(T, \{S_T, 0\}) = 0$.
The solution of this equation is

$$\tilde{\mathcal{C}} = \frac{\eta\,x^2}{T-t} \tag{3}$$

The optimal control is then

$$v_t^* = \frac{\partial_x \tilde{\mathcal{C}}}{2\,\eta} = \frac{x_t}{T-t}$$

It is optimal to liquidate stock at a constant rate v_t^* *independent* of the stock price S_t; the static VWAP strategy is dynamically optimal.

The statically optimal strategy

The statically optimal strategy v_s is the one that minimizes the cost function

$$\mathcal{C} = \mathbb{E}\left[\int_0^T \tilde{S}_s\, v_s\, ds\right] = \mathbb{E}\left[\int_0^T (S_s + \eta\, v_s)\, v_s\, ds\right] = \eta \int_0^T v_s^2\, ds$$

again with $v_s = -\dot{x}_s$.

The Euler-Lagrange equation is then

$$\partial_s v_s = -\partial_{s,s} x_s = 0$$

with boundary conditions $x_0 = X$ and $x_T = 0$ and the solution is obviously

$$v_t = \frac{X}{T};\; x_t = X\left(1 - \frac{t}{T}\right)$$

Remark

- This example suggests that we should always see if the static strategy is dynamically optimal rather than solve HJB directly.
  - Solving the Euler-Lagrange equation is much easier than solving the HJB equation!

Adding a risk term

In their paper [?], Almgren and Chriss added a risk term that penalized the variance of the trading cost.

$$\mathrm{Var}[\mathcal{C}] = \mathrm{Var}\left[\int_0^T x_t \, dS_t\right] = \sigma^2 \int_0^T x_t^2 \, dt$$

The expected risk-adjusted cost of trading was then given by

$$\mathcal{C} = \eta \int_0^T \dot{x}_t^2 \, dt + \lambda \, \sigma^2 \int_0^T x_t^2 \, dt$$

for some price of risk λ.

- Note the analogies to physics and portfolio theory.
  - The first term looks like kinetic energy and the second term like potential energy.
  - The expression looks like the objective in mean-variance portfolio optimization.

The Euler-Lagrange equation becomes

$$\ddot{x} - \kappa^2 x = 0$$

with

$$\kappa^2 = \frac{\lambda \sigma^2}{\eta}$$

The solution is a linear combination of terms of the form $e^{\pm\kappa t}$ that satisfies the boundary conditions $x_0 = X$, $x_T = 0$. The solution is then

$$x(t) = X \frac{\sinh \kappa(T - t)}{\sinh \kappa T}$$

Once again, it turns out that the statically optimal solution is dynamically optimal.

Brute force verification that the static solution is dynamically optimal

The state vector is $\{S_t, x_t\}$. Does the static solution satisfy the HJB equation? HJB in the risk-adjusted case reads:

$$C_t + \frac{1}{2}\,\sigma^2\,C_{S,S} + \lambda\,\sigma^2\,x_t^2 + \min_{v \in \mathcal{G}} \left\{-C_x\,v_t + (S_t + \eta\,v_t)\,v_t\right\} = 0 \quad (4)$$

and the optimal choice of v_t is again

$$v_t^* = \frac{1}{2\,\eta}\,(C_x - S_t)$$

Substituting back into (**??**) and defining $\tilde{C} := C - x\,S$ gives the equation for the cost function:

$$\tilde{C}_t = \frac{1}{4\,\eta}\,\left(\tilde{C}_x\right)^2 - \lambda\,\sigma^2\,x_t^2 \quad (5)$$

The cost

$$C = \eta \int_t^T ds \, \left\{ \dot{x}_s^2 + \kappa^2 \, x_s^2 \right\}$$

associated with the statically optimal trajectory

$$x_t = X \, \frac{\sinh \kappa \, (T - t)}{\sinh \kappa \, T}$$

is

$$C = \eta \, \kappa^2 \, (T - t) \, x_t^2 \, \frac{\cosh 2 \, \kappa \, (T - t)}{\sinh^2 \kappa \, (T - t)}$$

Then C is of the form

$$C = \eta \, \kappa \, x_t^2 \, g(\kappa \, (T - t))$$

Substituting this form back into equation (**??**) gives

$$-g'(\kappa\,(T-t)) = g(\kappa\,(T-t))^2 - 1$$

It is easy to verify that

$$g(\tau) = \frac{\cosh 2\,\tau}{\sinh^2\tau}$$

satisfies this equation and we conclude that the statically optimal solution is dynamically optimal.

What happens if we change the risk term?

Suppose we penalize average VaR instead of variance. This choice of risk term has the particular benefit of being linear in the position size. The expected risk-adjusted cost of trading is then given by

$$C = \eta \int_0^T \dot{x}_t^2 \, dt + \lambda \, \sigma \int_0^T x_t \, dt$$

for some price of risk λ.

The Euler-Lagrange equation becomes

$$\ddot{x} - A = 0$$

with

$$A = \frac{\lambda \, \sigma}{2 \, \eta}$$

The solution is a quadratic of the form $A\,t^2/2 + B\,t + C$ that satisfies the boundary conditions $x_0 = X$, $x_T = 0$. The solution is then

$$x(t) = \left(X - \frac{A\,T}{2}\,t\right)\left(1 - \frac{t}{T}\right) \tag{6}$$

In contrast to the previous case where the cost function is monotonic decreasing in the trading rate and the optimal choice of liquidation time is ∞, in this case, we can compute an optimal liquidation time. When T is optimal, we have

$$\frac{\partial C}{\partial T} \propto \dot{x}_T + A\,x_T = 0$$

from which we deduce that $\dot{x}_T = 0$.

Substituting into (**??**) and solving for the optimal time $T^\star$ gives

$$T^\star = \sqrt{\frac{2\,X}{A}}$$

With this optimal choice $T = T^\star$, the optimal strategy becomes

$$\begin{aligned} x(t) &= X\left(1-\frac{t}{T}\right)^2 \\ u(t) &= -\dot{x}(t) = 2\,X\left(1-\frac{t}{T}\right) \end{aligned}$$

One can verify that the static strategy is dynamically optimal, independent of the stock price.

An observation from Predoiu, Shaikhet and Shreve

Suppose the cost associated with a strategy depends on the stock price only through the term

$$\int_0^T S_t \, dx_t.$$

with S_t a martingale. Integration by parts gives

$$\mathbb{E}\left[\int_0^T S_t \, dx_t\right] = \mathbb{E}\left[S_T \, x_T - S_0 \, x_0 - \int_0^T x_t \, dS_t\right] = -S_0 \, X$$

which is independent of the trading strategy and we may proceed as if $S_t = 0$.

Quote from [?]

“...there is no longer a source of randomness in the problem. Consequently, without loss of generality we may restrict the search for an optimal strategy to nonrandom functions of time”.

Corollary

- This observation enables us to easily determine whether or not a statically optimal strategy will be dynamically optimal.
  - In particular, if the price process is of the form

    $$S_t = S_0 + \text{ impact of prior trading} + \text{noise},$$

    and if there is no risk term, a statically optimal strategy will be dynamically optimal.
  - If there is a risk term independent of the current stock price, a statically optimal strategy will again be dynamically optimal.

Forsyth et al.

- In [?], Forsyth et al. solve the HJB equation numerically under geometric Brownian motion with variance as the risk term so that the (random) cost is given by

$$\mathcal{C} = \eta \int_0^T \dot{x}_t^2 \, dt + \lambda \, \sigma^2 \int_0^T S_t^2 \, x_t^2 \, dt$$

- The efficient frontier is found to be virtually identical to the frontier computed in the arithmetic Brownian motion case.
- The problem of finding the optimal strategy is ill-posed; many strategies lead to almost the same value of the cost function.
- It is optimal to trade faster when the stock price is high so as to reduce variance. The optimal strategy is aggressive-in-the-money when selling stock and passive-in-the-money when buying stock.

Practical comments

- It's not clear what the price of risk should be.
- More often that not, a trader wishes to complete an execution before some final time and otherwise just wants to minimize expected execution cost.
  - In Almgren-Chriss style models, the optimal strategy is just VWAP (trading at constant rate).
- From now on, we will drop the risk term and the dynamics we will consider will ensure that the statically optimal solution is dynamically optimal.

The Almgren 2005 model

In this model [?], the stock price S_t evolves as

$$dS_t = \gamma\, dx_t + \sigma\, dZ_t$$

and the price $\tilde{S}_t$ at which we transact is given by

$$\tilde{S}_t = S_t + \eta\, v_t^{\delta}$$

where $v_t := -\dot{x}_t$ is the rate of trading.

The expected cost of trading is then given by

$$
\begin{aligned}
\mathcal{C} &= \mathbb{E}\left[\int_0^T \tilde{S}_t \, v_t \, dt\right] \\
&= \int_0^T (\gamma \, x_t + \eta \, v_t^{\delta}) \, v_t \, dt \\
&= \gamma \, (x_T^2 - x_0^2) + \eta \int_0^T v_t^{1+\delta} \, dt
\end{aligned}
$$

where wlog, we have set $S_0 = 0$.

We see that the first term corresponding to permanent impact is independent of the trading strategy, as it should be. The second term is convex in the trading rate so the minimum cost strategy is again VWAP.

Obizhaeva and Wang 2005

Recall from [?] that in this model,

$$S_t = S_0 + \eta \int_0^t u_s\, e^{-\rho\,(t-s)}\, ds + \int_0^t \sigma\, dZ_s \tag{7}$$

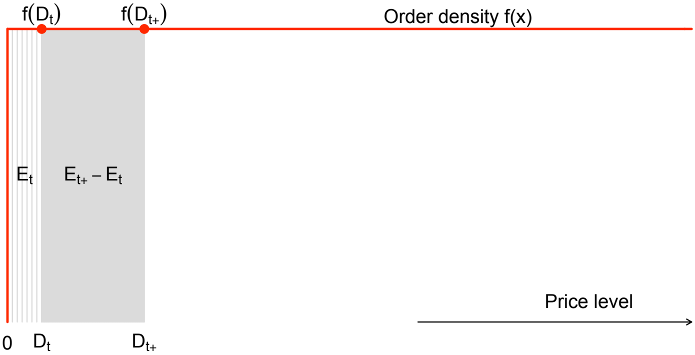
with $u_t = -\dot{x}_t$.

Market impact decays exponentially and instantaneous market impact is linear in the rate of trading.

The expected cost of trading becomes:

$$\mathcal{C} = \eta \int_0^T u_t\, dt \int_0^t u_s\, \exp\left\{-\rho\,(t-s)\right\}\, ds$$

Obizhaeva Wang order book process



When a trade of size ξ is placed at time t,

$$
\begin{aligned}
E_t &\mapsto E_{t+} = E_t + \xi \\
D_t = \eta\, E_t &\mapsto D_{t+} = \eta\, E_{t+} = \eta\,(E_t + \xi)
\end{aligned}
$$

When the trading policy is statically optimal, the Euler-Lagrange equation applies:

$$\frac{\partial}{\partial t}\frac{\delta \mathcal{C}}{\delta u_t} = 0$$

where $u_t = \dot{x}_t$. Functionally differentiating $\mathcal{C}$ with respect to u_t gives

$$\frac{\delta \mathcal{C}}{\delta u_t} = \int_0^t u_s\, e^{-\rho\,(t-s)}\, ds + \int_t^T u_s\, e^{-\rho\,(s-t)}\, ds = A \tag{8}$$

for some constant A. Equation (**??**) may be rewritten as

$$\int_0^T u_s\, e^{-\rho\,|t-s|}\, ds = A$$

which is a Fredholm integral equation of the first kind (see [**?**]).

Now substitute

$$u_s = \delta(s) + \rho + \delta(s - T)$$

into (**??**) to obtain

$$\frac{\delta \mathcal{C}}{\delta u_t} = e^{-\rho\, t} + \left(1 - e^{-\rho\, t}\right) = 1$$

The optimal strategy consists of a block trade at time $t = 0$, continuous trading at the rate ρ over the interval $(0, T)$ and another block trade at time $t = T$.

Consider the volume impact process E_t. The initial block-trade causes

$$0 = E_0 \mapsto E_{0+} = 1$$

According to the assumptions of the model, the volume impact process reverts exponentially so

$$E_t = E_{0+}\, e^{-\rho\, t} + \rho \int_0^t e^{-\rho\,(t-s)}\, ds = 1$$

i.e. the volume impact process is constant when the trading strategy is optimal.

Optimality

- Originally, Obizhaeva and Wang[**?**] derived their solution in discrete time by explicitly solving the HJB equation.
- Predoiu, Shaikhet and Shreve's[**?**] observation allows us to deduce that the statically optimal solution we just derived is also dynamically optimal.

The model of Alfonsi, Fruth and Schied

Alfonsi, Fruth and Schied [?] consider the following (AS) model of the order book:

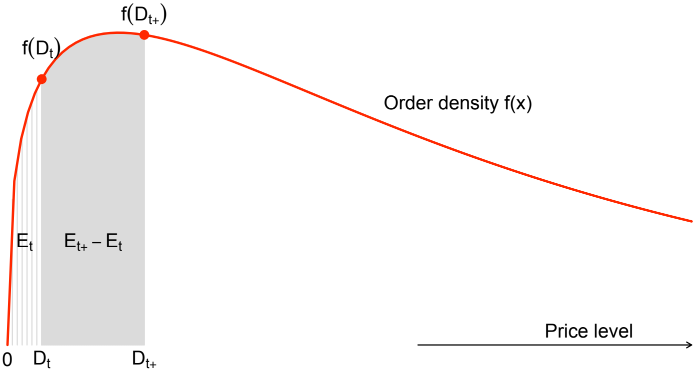
- There is a continuous (in general nonlinear) density of orders $f(x)$ above some martingale ask price A_t. The cumulative density of orders up to price level x is given by

$$F(x) := \int_0^x f(y)\, dy$$

- Executions eat into the order book (*i.e.* executions are with market orders).
- A purchase of ξ shares at time t causes the ask price to increase from $A_t + D_t$ to $A_t + D_{t+}$ with

$$\xi = \int_{D_t}^{D_{t+}} f(x)\, dx = F(D_{t+}) - F(D_t)$$

Schematic of the model



When a trade of size ξ is placed at time t,

$$\begin{aligned} E_t &\mapsto E_{t+} = E_t + \xi \\ D_t = F^{-1}(E_t) &\mapsto D_{t+} = F^{-1}(E_{t+}) = F^{-1}(E_t + \xi) \end{aligned}$$

Optimal liquidation strategy in the AS model

The cost of trade execution in the AS model is given by:

$$\mathcal{C} = \int_0^T v_t \, F^{-1}(E_t) \, dt + \sum_{t \leq T} \left[H(E_{t+}) - H(E_t) \right] \tag{9}$$

where

$$E_t = \int_0^t u_s \, e^{-\rho\,(t-s)} \, ds$$

is the volume impact process and

$$H(x) = \int_0^x F^{-1}(x) \, dx$$

gives the cost of executing an instantaneous block trade of size x.

Consider the ansatz $u_t = \xi_0\, \delta(t) + \xi_0\, \rho + \xi_T \delta(T - t)$. For $t \in (0, T)$, we have $E_t = E_0 = \xi_0$, a constant. With this choice of u_t, we would have

$$\begin{aligned}
\mathcal{C}(X) &= F^{-1}(\xi_0) \int_0^T v_t\, dt + [H(E_{0^+}) - H(E_0)] + [H(E_T) - H(E_{T^-})] \\
&= F^{-1}(\xi_0)\, \xi_0\, \rho\, T + H(\xi_0) + [H(\xi_0 + \xi_T) - H(\xi_0)] \\
&= F^{-1}(\xi_0)\, \xi_0\, \rho\, T + H(X - \rho\, \xi_0\, T)
\end{aligned}$$

Differentiating this last expression gives us the condition satisfied by the optimal choice of ξ_0:

$$F^{-1}(X - \rho\, \xi_0\, T) = F^{-1}(\xi_0) + {F^{-1}}'(\xi_0)\, \xi_0$$

or equivalently

$$F^{-1}(\xi_0 + \xi_T) = F^{-1}(\xi_0) + {F^{-1}}'(\xi_0)\, \xi_0$$

Functionally differentiating $\mathcal{C}$ with respect to u_t gives

$$\begin{aligned} \frac{\delta \mathcal{C}}{\delta u_t} &= F^{-1}(E_t) + \int_t^T u_s \, F^{-1\prime}(E_s) \, \frac{\delta E_s}{\delta u_t} \, ds \\ &= F^{-1}(E_t) + \int_t^T u_s \, F^{-1\prime}(E_s) \, e^{-\rho\,(s-t)} \, ds \qquad (10) \end{aligned}$$

The first term in (**??**) represents the marginal cost of new quantity at time t and the second term represents the marginal extra cost of future trading.

With our ansatz, and a careful limiting argument, we obtain

$$\begin{aligned} \frac{\delta \mathcal{C}}{\delta u_t} &= F^{-1}(\xi_0) + \xi_0 \, F^{-1\prime}(\xi_0) \left[1 - e^{-\rho\,(T-t)}\right] \\ &\quad + e^{-\rho\,(T-t)} \left[F^{-1}(\xi_T + \xi_0) - F^{-1}(\xi_0)\right] \end{aligned}$$

Imposing our earlier condition on ξ_T gives

$$\begin{aligned}\frac{\delta\mathcal{C}}{\delta u_t} &= F^{-1}(\xi_0) + \xi_0\, F^{-1\prime}(\xi_0)\,\left[1 - e^{-\rho\,(T-t)}\right] \\ &\quad + e^{-\rho\,(T-t)}\,\xi_0\, F^{-1\prime}(\xi_0) \\ &= F^{-1}(\xi_0) + \xi_0\, F^{-1\prime}(\xi_0)\end{aligned}$$

which is constant, demonstrating (static) optimality.

Example

With $F^{-1}(x) = \sqrt{x}$,

$$\sqrt{\xi_0 + \xi_T} = F^{-1}(\xi_0 + \xi_T) = F^{-1}(\xi_0) + F^{-1\prime}(\xi_0)\,\xi_0 = \sqrt{\xi_0} + \frac{1}{2}\sqrt{\xi_0}$$

which has the solution $\xi_T = \frac{5}{4}\,\xi_0$.

Generalization

Alexander Weiss [?] and then Predoiu, Shaikhet and Shreve [?] have shown that the bucket-shaped strategy is optimal under more general conditions than exponential resiliency. Specifically, if resiliency is a function of E_t (or equivalently D_t) only, the optimal strategy has a block trades at inception and completion and continuous trading at a constant rate in-between.

Optimality and price manipulation

- For all of the models considered so far, there was an optimal strategy.
- The optimal strategy always involved trades of the same sign. So no sells in a buy program, no buys in a sell program.
- It turns out (see [?]) that we can write down models for which price manipulation is possible.
  - For example, square root price impact with exponential decay admits price manipulation.
- In such cases, a round-trip trade can generate cash on average.
  - You would want to repeat such a trade over and over.
  - There would be no optimal strategy.

Linear transient market impact

The price process assumed in [?] is

$$S_t = S_0 + \int_0^t h(v_s)\, G(t-s)\, ds + \text{ noise}$$

In [6], this model is on the one hand extended to explicitly include discrete optimal strategies and on the other hand restricted to the case of linear market impact. When the admissible strategy X is used, the price S_t is given by

$$S_t = S_t^0 + \int_{\{s<t\}} G(t-s)\, dX_s, \tag{11}$$

and the expected cost of liquidation is given by

$$\mathcal{C}(X) := \frac{1}{2} \int \int G(|t-s|)\, dX_s\, dX_t. \tag{12}$$

Condition for no price manipulation

Definition (Huberman and Stanzl)

A *round trip* is an admissible strategy with $X_0 = 0$. A *price manipulation strategy* is a round trip with strictly negative expected costs.

Proposition (Bochner)

$\mathcal{C}(X) \geq 0$ for all admissible strategies X if and only if $G(|\cdot|)$ can be represented as the Fourier transform of a positive finite Borel measure μ on $\mathbb{R}$, i.e.,

$$G(|x|) = \int e^{ixz}\,\mu(dz).$$

First order condition

Theorem

Suppose that G is positive definite. Then X^* minimizes $\mathcal{C}(\cdot)$ if and only if there is a constant λ such that X^* solves the generalized Fredholm integral equation

$$\int G(|t-s|)\, dX^*_s = \lambda \qquad \text{for all } t \in \mathbb{T}. \tag{13}$$

In this case, $\mathcal{C}(X^*) = \frac{1}{2}\lambda x$. In particular, λ must be nonzero as soon as G is strictly positive definite and $x \neq 0$.

Transaction-triggered price manipulation

Definition (Alfonsi, Schied, Slynko (2009))

A market impact model admits *transaction-triggered price manipulation* if the expected costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

As discussed in [4], transaction-triggered price manipulation can be regarded as an additional model irregularity that should be excluded. Transaction-triggered price manipulation can exist in models that do not admit standard price manipulation in the sense of Huberman and Stanzl definition.

Condition for no transaction-triggered price manipulation

Theorem

Suppose that the decay kernel $G(\cdot)$ is convex, satisfies $\int_0^1 G(t)\,dt < \infty$ and that the set of admissible strategies is nonempty. Then there exists a unique admissible optimal strategy X^*. Moreover, X_t^* is a monotone function of t, and so there is no transaction-triggered price manipulation.

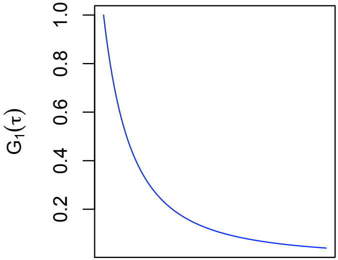
Remark

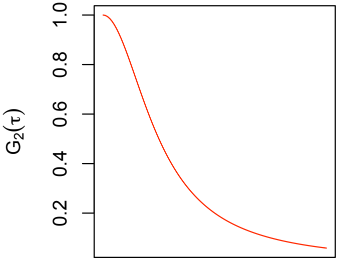
If G is not convex in a neighborhood of zero, there is transaction-triggered price manipulation.

An instructive example

We solve a discretized version of the Fredholm equation (with 512 time points) for two similar decay kernels:

$$G_1(\tau) = \frac{1}{(1+t)^2}; \qquad G_2(\tau) = \frac{1}{1+t^2}$$

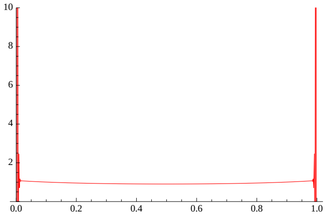




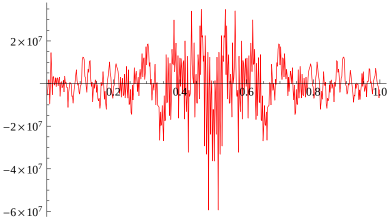
$G_1(\cdot)$ is convex, but $G_2(\cdot)$ is concave near $\tau = 0$ so there should be a unique optimal strategy with $G_1(\cdot)$ as a choice of kernel but there should be transaction-triggered price manipulation with $G_2(\cdot)$ as the choice of decay kernel.

Schematic of numerical solutions of Fredholm equation

$G_1(\tau) = \frac{1}{(1+t)^2}$



$G_2(\tau) = \frac{1}{1+t^2}$



In the left hand figure, we observe block trades at $t = 0$ and $t = 1$ with continuous (nonconstant) trading in $(0, 1)$. In the right hand figure, we see numerical evidence that the optimal strategy does not exist.

Now we give some examples of the optimal strategy with choices of kernel that preclude transaction-triggered price manipulation.

Example I: Linear market impact with exponential decay

$G(\tau) = e^{-\rho\,\tau}$ and the optimal strategy $u(s)$ solves

$$\int_0^T u(s) e^{-\rho\,|t-s|}\,ds = \text{const.}$$

We already derived the solution which is

$$u(s) = A\,\{\delta(t) + \rho + \delta(T-t)\}$$

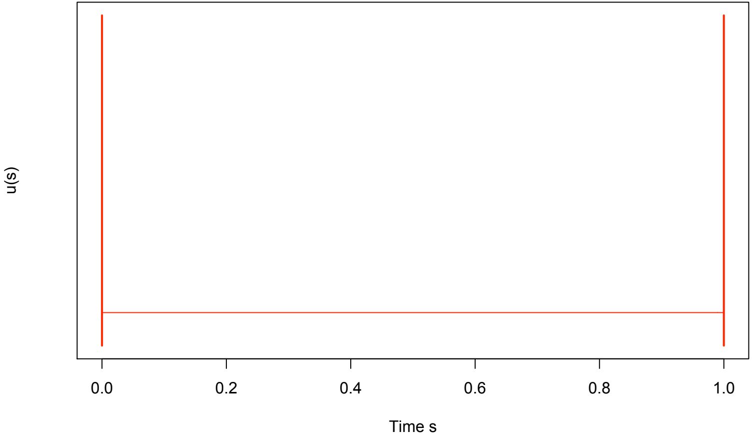
The normalizing factor A is given by

$$\int_0^T u(t)\,dt = X = A\,(2 + \rho\,T)$$

The optimal strategy consists of block trades at $t = 0$ and $t = T$ and continuous trading at the constant rate ρ between these two times.

Schematic of optimal strategy

The optimal strategy with $\rho = 0.1$ and $T = 1$



Example II: Linear market impact with power-law decay

$G(\tau) = \tau^{-\gamma}$ and the optimal strategy $u(s)$ solves

$$\int_0^T \frac{u(s)}{|t-s|^\gamma}\, ds = \text{const.}$$

The solution is

$$u(s) = \frac{A}{[s\,(T-s)]^{(1-\gamma)/2}}$$

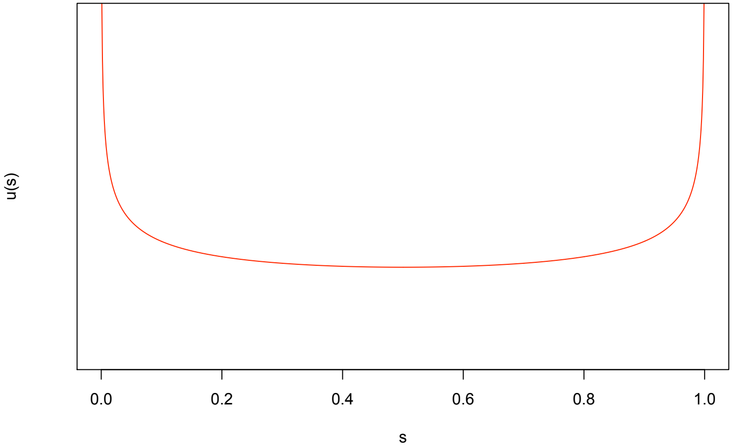
The normalizing factor A is given by

$$\int_0^T u(t)\,dt = X = A\sqrt{\pi}\,\left(\frac{T}{2}\right)^\gamma \frac{\Gamma\left(\frac{1+\gamma}{2}\right)}{\Gamma\left(1+\frac{\gamma}{2}\right)}$$

The optimal strategy is absolutely continuous with no block trades. However, it is singular at $t=0$ and $t=T$.

Schematic of optimal strategy

The red line is a plot of the optimal strategy with $T = 1$ and $\gamma = 1/2$.



Example III: Linear market impact with linear decay

$G(\tau) = (1 - \rho\,\tau)^+$ and the optimal strategy $u(s)$ solves

$$\int_0^T u(s)\,(1 - \rho\,|t - s|)^+\,ds = \text{const.}$$

Let $N := \lfloor \rho\,T \rfloor$, the largest integer less than or equal to $\rho\,T$. Then

$$u(s) = A \sum_{i=0}^{N} \left(1 - \frac{i}{N+1}\right) \left\{ \delta\left(s - \frac{i}{\rho}\right) + \delta\left(T - s - \frac{i}{\rho}\right) \right\}$$

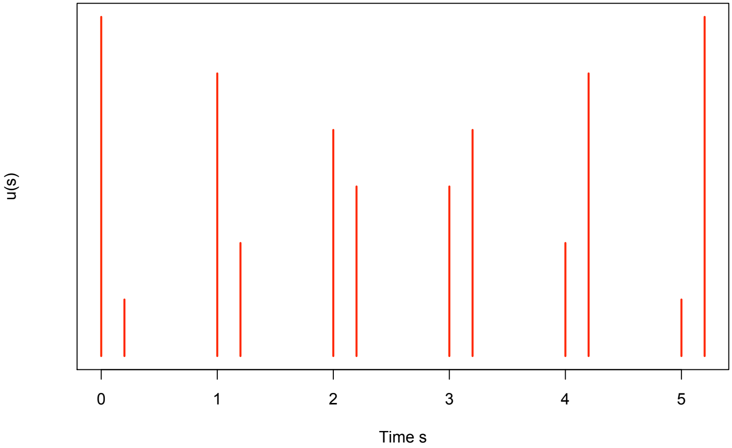
The normalizing factor A is given by

$$\int_0^T u(t)\,dt = X = A \sum_{i=0}^{N} 2\left(1 - \frac{i}{N+1}\right) = A\,(2 + N)$$

The optimal strategy consists only of block trades with no trading between blocks.

Schematic of optimal strategy

Positions and relative sizes of the block trades in the optimal strategy with $\rho = 1$ and $T = 5.2$ (so $N = \lfloor \rho T \rfloor = 5$).



Summary I

- The optimal trading strategy depends on the model.
  - For Almgren-Chriss style models, if the price of risk is zero, the minimal cost strategy is VWAP.
  - In Alfonsi-Schied style models with resiliency that depends only on the current spread, the minimal cost strategy is to trade a block at inception, a block at completion and at a constant rate in between.
  - We exhibited other models for which the optimal strategy is more interesting.
- In most conventional models, the optimal liquidation strategy is independent of the stock price.
  - However, for each such model, it is straightforward to specify a similar model in which the optimal strategy does depend on the stock price.

Summary II

- In some models, price manipulation is possible and there is no optimal strategy.
- It turns out that we also need to exclude *transaction-triggered price manipulation*.
  - We presented example of models for which price manipulation is possible.
  - In the case of linear transient impact, we provided conditions under which transaction-triggered price manipulation is precluded.