Rational Shapes of the Volatility Surface

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Goals

- Derive arbitrage bounds on the slope and curvature of volatility skews.
- Investigate the strike and time behavior of these bounds.
- Specialize to stochastic volatility and jumps.
- Draw implications for parameterization of the volatility surface.

Slope Constraints

- No arbitrage implies that call spreads and put spreads must be non-negative. *i.e.*

$$\frac{\partial C}{\partial K} \leq 0 \text{ and } \frac{\partial P}{\partial K} \geq 0$$

- In fact, we can tighten this to

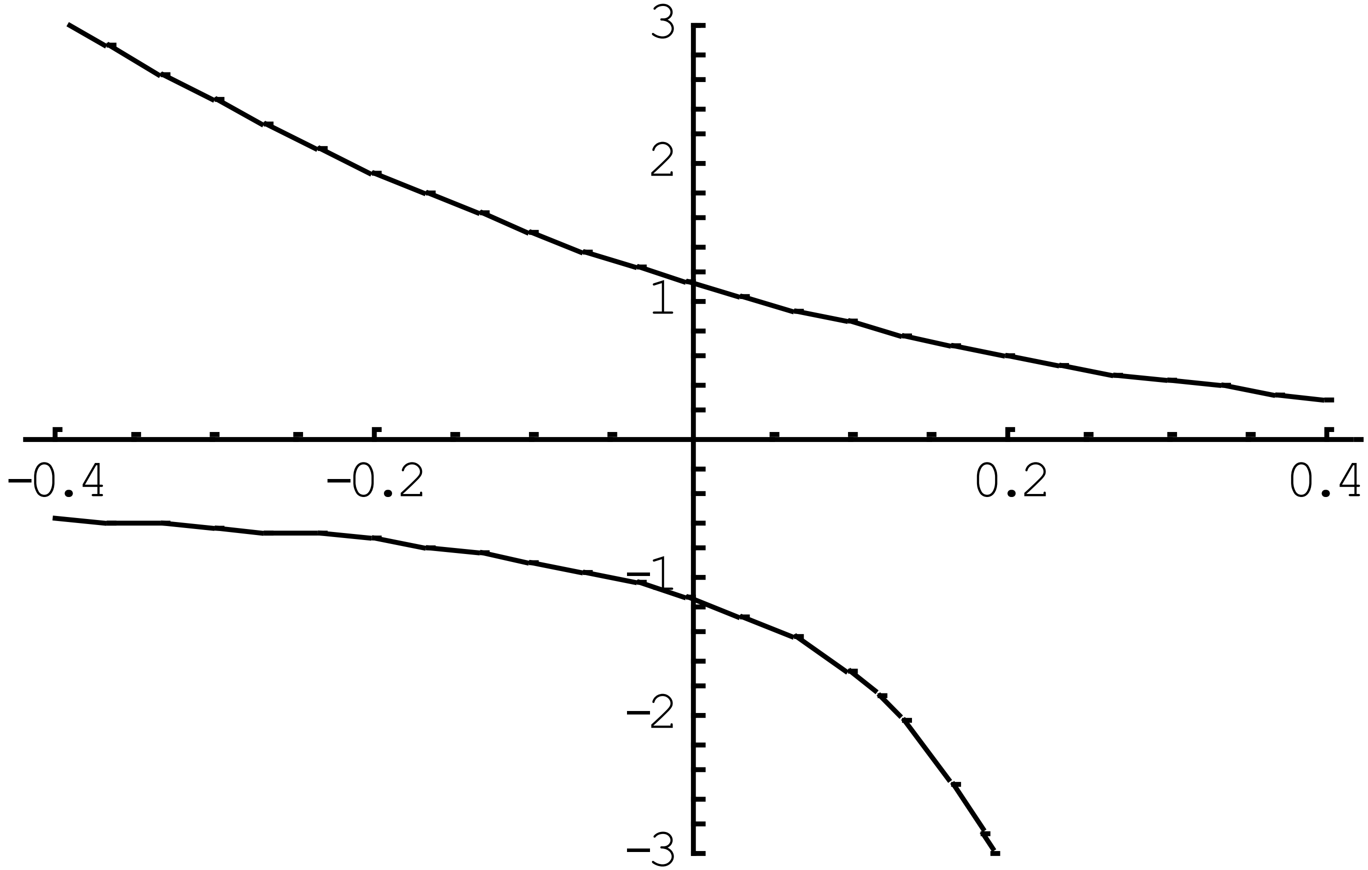
$$\frac{\partial C}{\partial K} \leq 0 \text{ and } \frac{\partial}{\partial K}\left(\frac{P}{K}\right) \geq 0$$

- Translate these equations into conditions on the implied total volatility $\sigma[y]$ as a function of $y = \ln\left(K/F\right)$.
- In conventional notation, we get

$$\sigma'[y] \le \sqrt{2\pi}\,\exp\left\{d_2^{\,2}/2\right\} N\left(d_2\right)$$

$$\sigma'[y] \ge -\sqrt{2\pi}\,\exp\left\{d_1^{\,2}/2\right\} N\left(-d_1\right)$$

- Assuming $\sigma[y] = 0.25 - 0.3y$ we can plot these bounds on the slope as functions of y .



- Note that we have plotted bounds on the slope of *total* implied volatility as a function of y. This means that the bounds on the slope of BS implied volatility get tighter as time to expiration increases by $1/\sqrt{T}$.

Convexity Constraints

- No arbitrage implies that call and puts must have positive convexity. *i.e.*

$$\frac{\partial^2 C}{\partial K^2} \geq 0 \text{ and } \frac{\partial^2 P}{\partial K^2} \geq 0$$

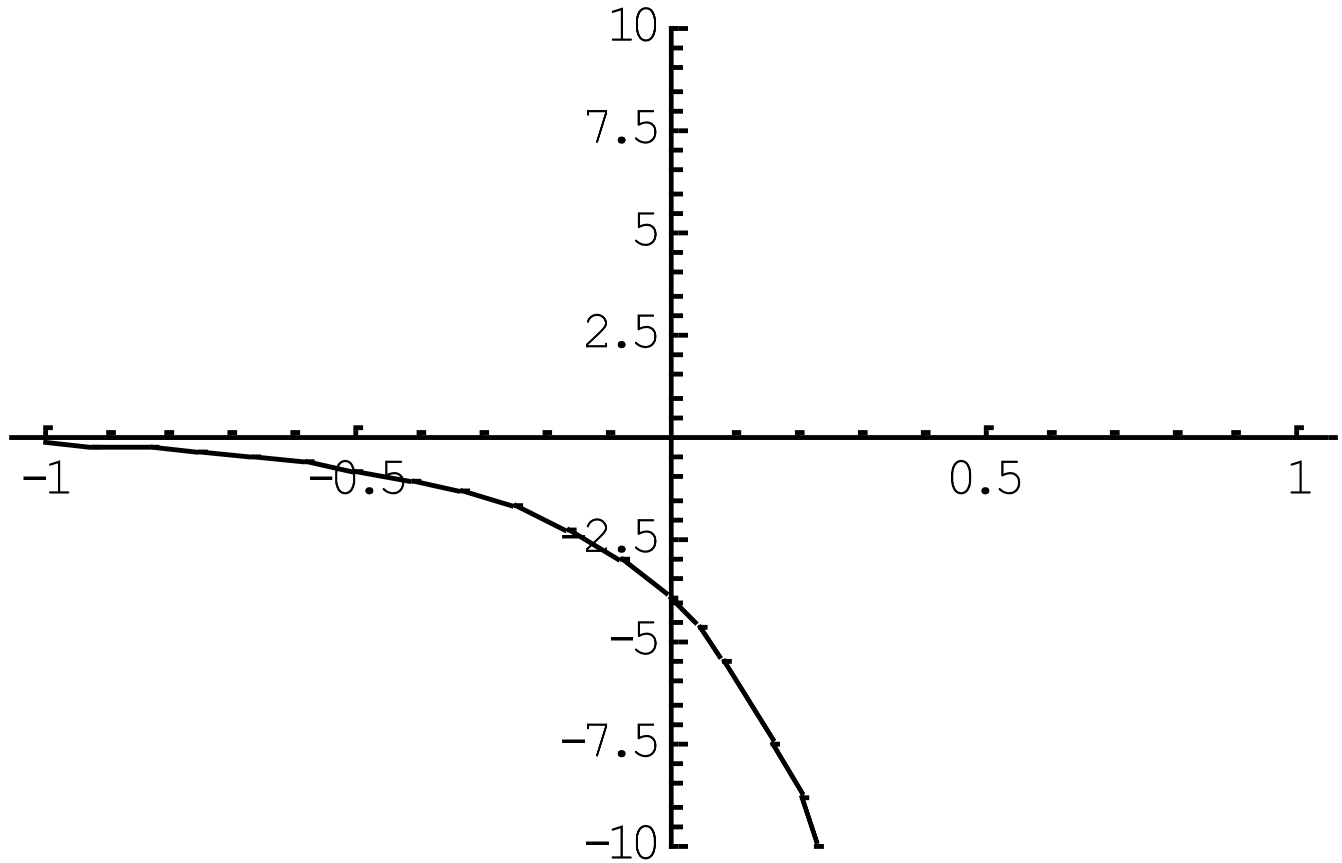
- Translating these into our variables gives

$$\frac{\partial^2 C}{\partial y^2} \geq \frac{\partial C}{\partial y}$$

- We get a complicated expression which is nevertheless easy to evaluate for any particular function $\sigma[y]$.

- This expression is equivalent to demanding that butterflies have non-negative value.

- Again, assuming $\sigma[y] = 0.25$ and $\sigma'[y] = -0.3$ we can plot this lower bound on the convexity as a function of y.



Implication for Variance Skew

- Putting together the vertical spread and convexity conditions, it may be shown that implied variance may not grow faster than linearly with the log-strike.
- Formally,

$$\frac{v[y]}{y} \equiv \frac{\sigma_{BS}{}^{2}[y]}{y} \rightarrow \text{ some constant } A \text{ as } |\mathrm{y}| \rightarrow \infty$$

Local Volatility

- Local volatility $\sigma(K,T)$ is given by

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C}{\partial T}}{K^2 \dfrac{\partial^2 C}{\partial K^2}}$$

- Local variances are non-negative iff arbitrage constraints are satisfied.

Time Behavior of the Skew

- Since in practice, we are interested in the lower bound on the slope for most stocks, let's investigate the time behavior of this lower bound.
- Recall that the lower bound on the slope can be expressed as

$$-\sqrt{2\pi}\exp\left(-d_1^{\,2}/2\right)N\left(-d_1\right)$$

- For small times, $d_1 \approx 0$ and $N\left(d_1\right) \approx \frac{1}{2}$

  so

$$\sigma'[0] \geq -\sqrt{\frac{\pi}{2}}$$

  Reinstating explicit dependence on T, we get

$$\sigma_{BS}'[0] \geq -\sqrt{\frac{\pi}{2\,T}}$$

  That is, $\sqrt{T}$ for small T.

- Also,

$$d_1 = \frac{\sigma[0]}{2} \to \infty \text{ as } t \to \infty$$

- Then, the lower bound on the slope

$$\sigma'[0] \geq -\sqrt{2\pi}\, \exp\left(d_1^{\,2}/2\right) N\left(-d_1\right)$$

$$\approx -\frac{1}{d_1} = -\frac{2}{\sigma[0]}$$

- Making the time-dependence of $\sigma[0]$ explicit,

$$\sigma_{BS}'[0] \geq -\frac{1}{T}\frac{2}{\sigma_{BS}[0]} \text{ as } T \to \infty$$

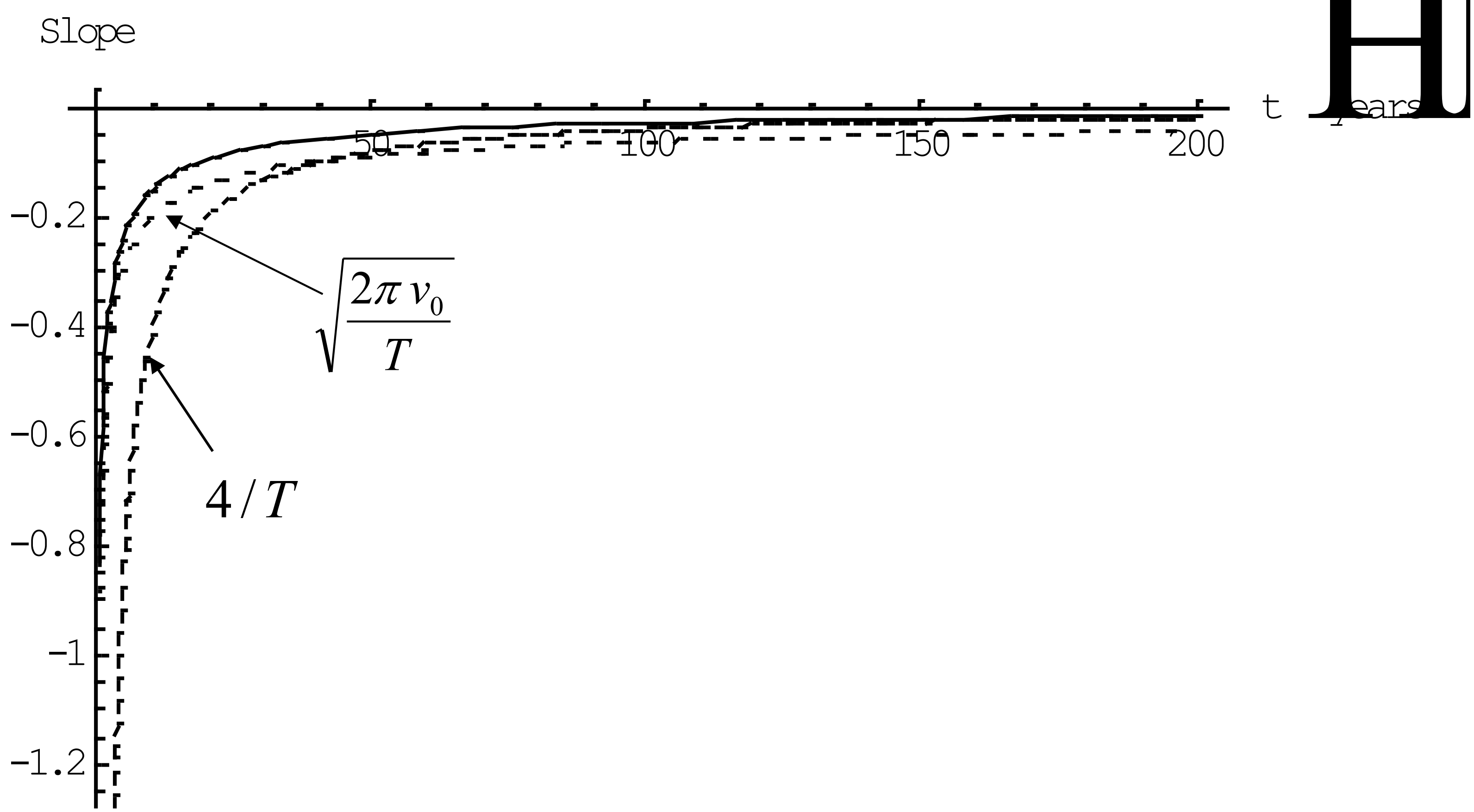
- In particular, the time dependence of the at-the-money skew cannot be

$$\sigma_{BS}'[0] \approx -\frac{1}{\sqrt{T}}$$

because for any choice of positive constants a, b

$$\exists T \text{ large enough s.t. } -\frac{a}{\sqrt{T}} < -\frac{b}{T}$$

- Assuming $\sigma_{BS}[0] = 0.25$, we can plot the variance slope lower bound as a function of time.



A Practical Example of Arbitrage

- We suppose that the ATMF 1 year volatility and skew are 25% and 11% per 10% respectively. Suppose that we extrapolate the vol skew using a $1/\sqrt{T}$ rule.
- Now, buy 99 puts struck at 101 and sell 101 puts struck at 99. What is the value of this portfolio as a function of time to expiration?

Current Market	100.00	100.00	100.00	100.00
Dividends (cts. yield or schedule)	0.00%	0.00%	0.00%	0.00%
Strike	101.00	99.00	101.00	99.00
Start Date (date on which strike is se	03-Apr-98	03-Apr-98	03-Apr-98	03-Apr-98
Shares = s, Notional = n	s	s	s	s
Expiration Date	03-Apr-99	03-Apr-99	03-Apr-02	03-Apr-02
Stock Rate (sa/365 rate or curve)	0.000%	0.000%	0.000%	0.000%
Pay Rate (sa/365 rate or curve)	0.000%	0.000%	0.000%	0.000%
Volatility (number or curve)	23.90%	26.10%	24.45%	25.55%
Call =c, Put= p	p	p	p	p
Option Price	**10.07**	**9.84**	**19.92**	**19.58**
Delta	**-0.4690**	**-0.4329**	**-0.4113**	**-0.3916**
Gamma (per 1%)	**0.0166**	**0.0151**	**0.0080**	**0.0075**
Vega per 1% vol	**0.3976**	**0.3932**	**0.7774**	**0.7675**
Theta per day	**-0.0130**	**-0.0141**	**-0.0065**	**-0.0067**
Position	**99**	**-101**	**99**	**-101**
Value	**996.72**	**(993.70)**	**1,972.34**	**(1,977.18)**
Portfolio Value	**3.02**		**(4.83)**	

Arbitrage!

With more reasonable parameters, it takes a long time to generate an arbitrage though….

Current Market	100.00	100.00	100.00	100.00
Dividends (cts. yield or schedule)	0.00%	0.00%	0.00%	0.00%
Strike	101.00	99.00	101.00	99.00
Start Date (date on which strike is se	03-Apr-98	03-Apr-98	03-Apr-98	03-Apr-98
Shares = s, Notional = n	s	s	s	s
Expiration Date	03-Apr-99	03-Apr-99	03-Apr-48	03-Apr-48
Stock Rate (sa/365 rate or curve)	0.000%	0.000%	0.000%	0.000%
Pay Rate (sa/365 rate or curve)	0.000%	0.000%	0.000%	0.000%
Volatility (number or curve)	24.70%	25.30%	24.96%	25.04%
Call =c, Put= p	p	p	p	p
Option Price	**10.39**	**9.52**	**63.07**	**61.61**
Delta	**-0.4668**	**-0.4340**	**-0.1902**	**-0.1864**
Gamma (per 1%)	**0.0161**	**0.0156**	**0.0015**	**0.0015**
Vega per 1% vol	**0.3975**	**0.3934**	**1.8909**	**1.8670**
Theta per day	**-0.0135**	**-0.0136**	**-0.0013**	**-0.0013**
Position	**99**	**-101**	**99**	**-101**
Value	**1,028.21**	**(961.92)**	**6,244.14**	**(6,222.68)**
Portfolio Value	**66.30**	**No arbitrage!** →	**21.46**	

50 Years!

So Far….

- We have derived arbitrage constraints on the slope and convexity of the volatility skew.
- We have demonstrated that the $1/\sqrt{T}$ rule for extrapolating the skew is inconsistent with no arbitrage. Time dependence must be at most $1/T$ for large T

Stochastic Volatility

- Consider the following special case of the Heston model:

$$dx = \mu\, dt + \sqrt{v}\, dZ$$

$$dv = -\lambda\left(v - \overline{v}\right) dt - \eta \sqrt{v}\, dZ$$

- In this model, it can be shown that

$$\left.\frac{\partial v_{BS}}{\partial y}\right|_{y=0} \approx -\eta \frac{1}{\lambda T}\left\{1 - \frac{1 - e^{-\lambda T}}{\lambda T}\right\}$$

- For a general stochastic volatility theory of the form:

$$dx = \mu\, dt + \sqrt{v}\, dZ_1$$

$$dv = -\lambda\left(v - \bar{v}\right) dt - \eta\, \beta\left(v\right) \sqrt{v}\, dZ_2$$

with

$$\left\langle dZ_1, dZ_2 \right\rangle = \rho\, dt$$

we claim that (very roughly)

$$\left.\frac{\partial v_{BS}}{\partial y}\right|_{y=0} \approx \rho\eta\, \beta\left(v\right) \frac{1}{\lambda T}\left\{1 - \frac{1-e^{-\lambda T}}{\lambda T}\right\}$$

- Then, for very short expirations, we get

$$\left.\frac{\partial v_{BS}}{\partial y}\right|_{y=0} \approx \frac{\rho\eta\,\beta(v)}{2}$$

- a result originally derived by Roger Lee and for very long expirations, we get

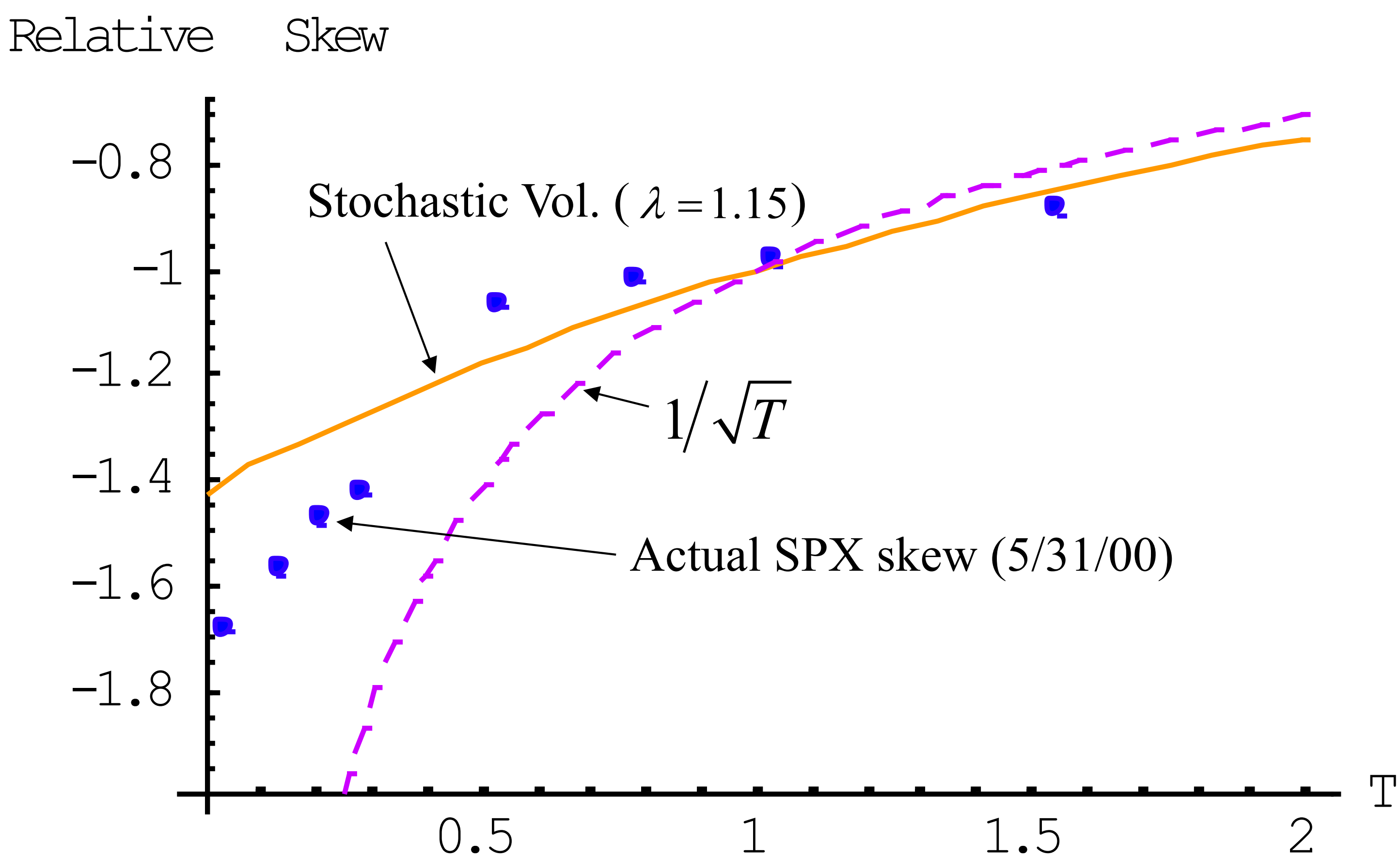
$$\left.\frac{\partial v_{BS}}{\partial y}\right|_{y=0} \approx \frac{\rho\eta\,\beta(v)}{\lambda T}$$

- Both of these results are consistent with the arbitrage bounds.

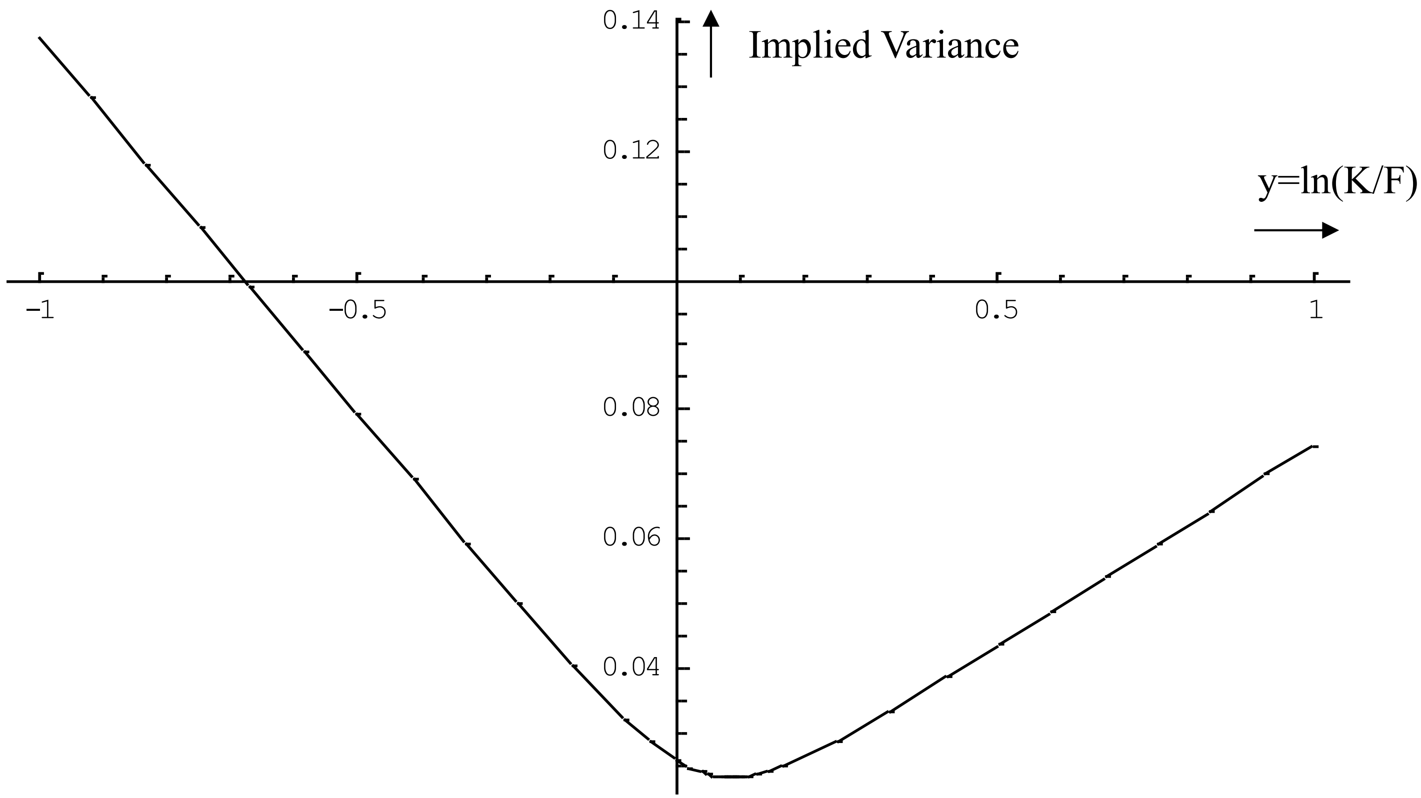
Doesn't This Contradict $\sqrt{T}$?

- Market practitioners' rule of thumb is that the skew decays as $1/\sqrt{T}$.
- Using $\lambda = 1.15$ (from Bakshi, Cao and Chen), we get the following graph for the relative size of the at-the-money variance skew:

ATM Skew as a Function of T



Heston Implied Variance



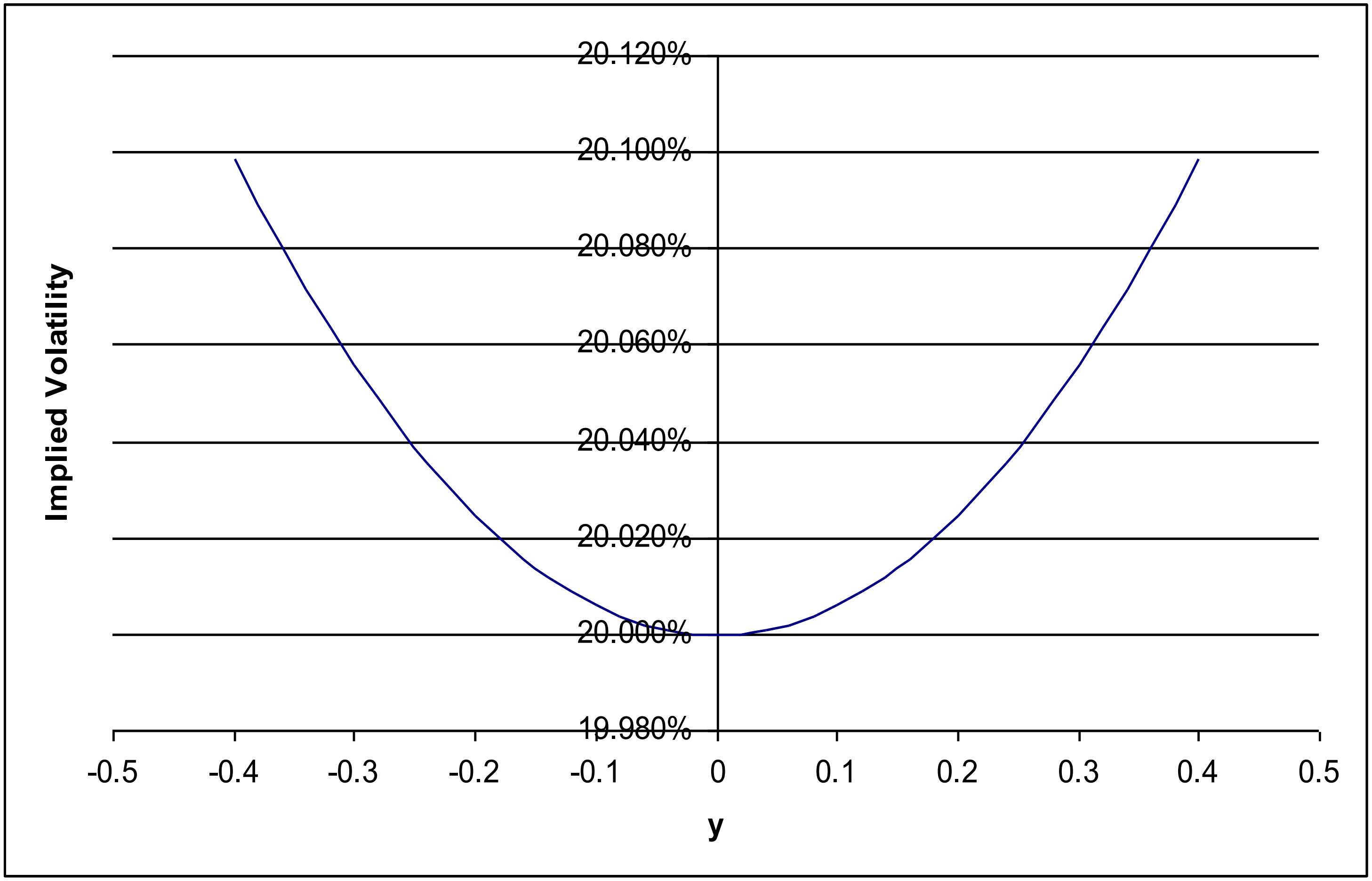
Parameters: $v = 0.04, \bar{v} = 0.04, \lambda = 1.15, \rho = \text{-}0.39, \eta = 0.64$

from Bakshi, Cao and Chen.

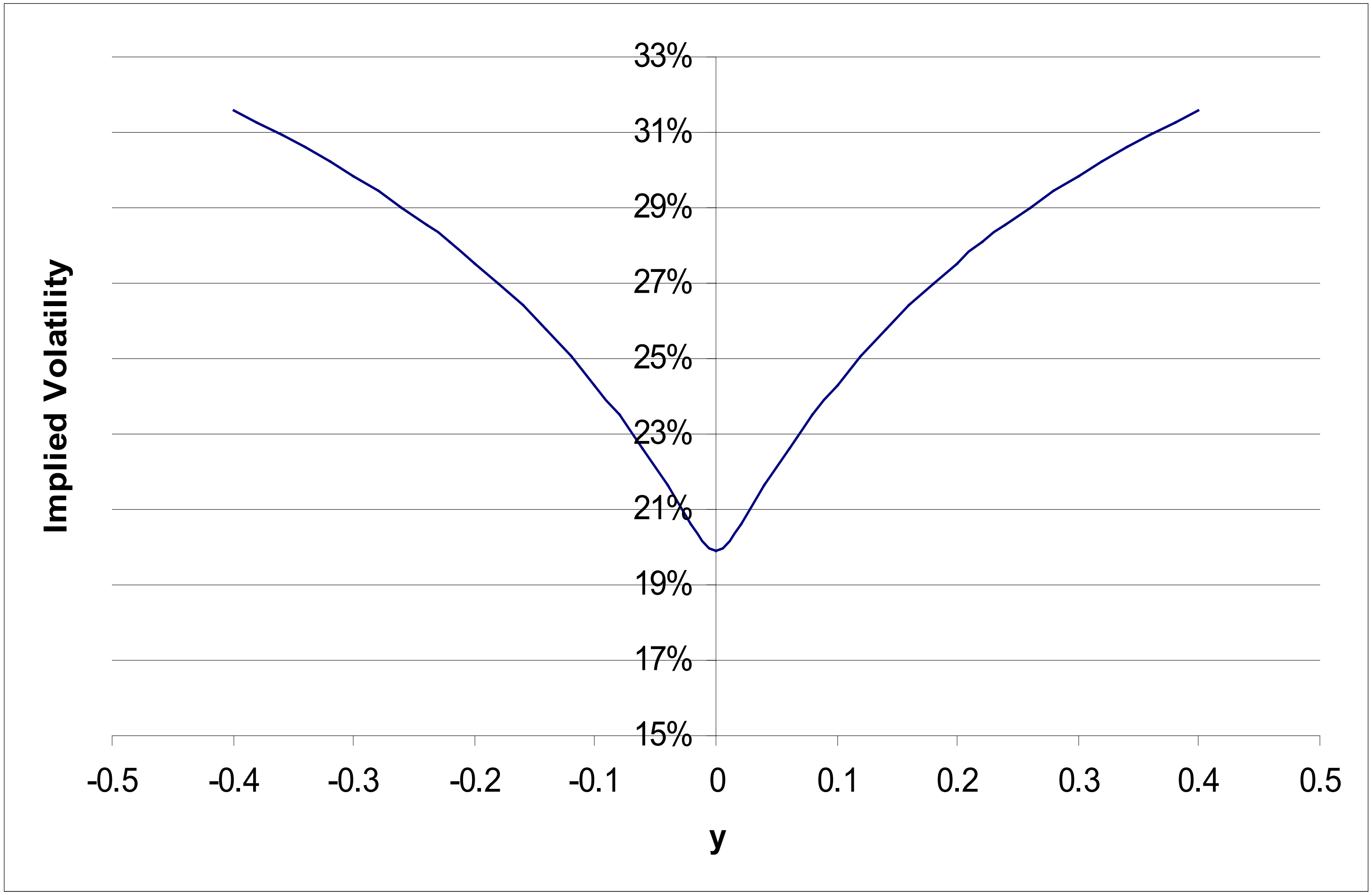
A Simple Regime Switching Model

- To get intuition for the impact of volatility convexity, we suppose that realised volatility over the life of a one year option can take one of two values each with probability 1/2. The average of these volatilities is 20%.
- The price of an option is just the average option price over the two scenarios.
- We graph the implied volatilities of the resulting option prices.

High Vol: 21%; Low Vol: 19%



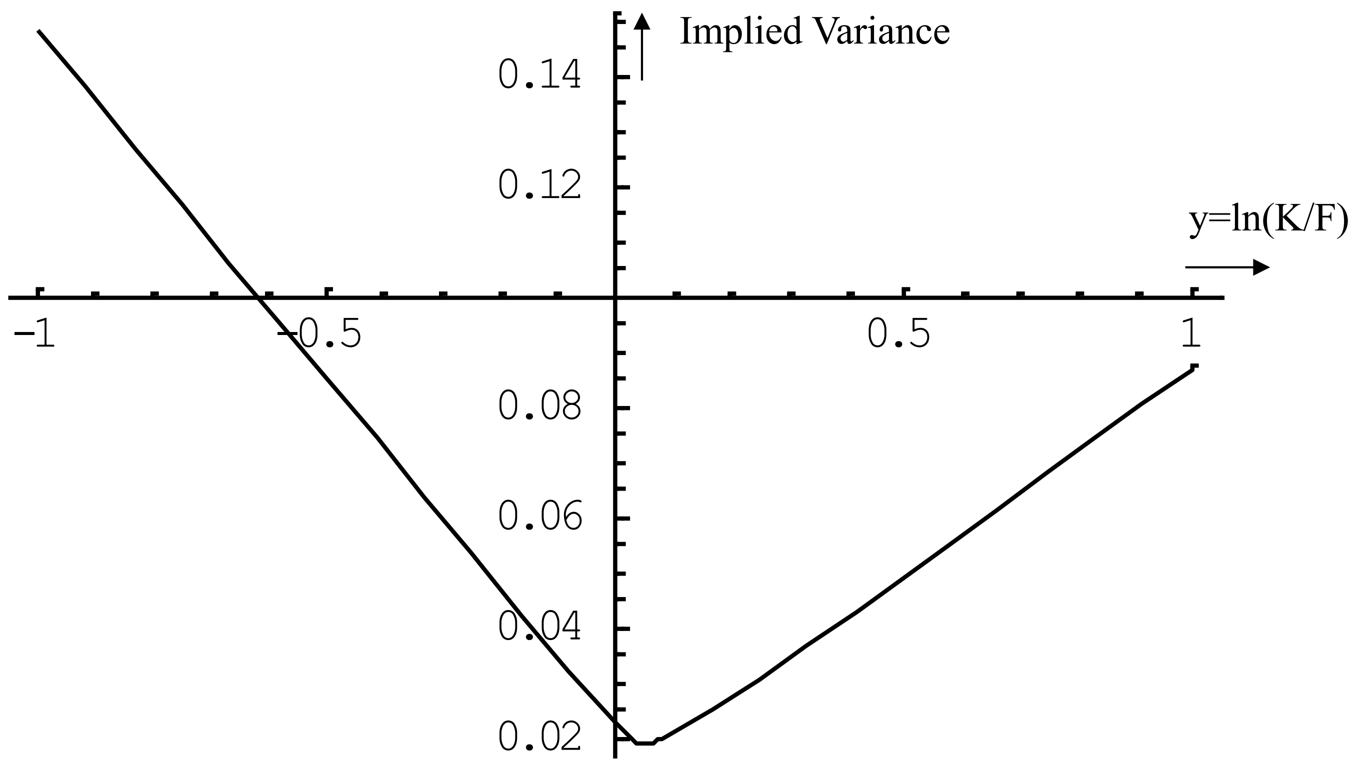
High Vol: 39%; Low Vol: 1%



Intuition

- As $\left| y \right| \to \infty$, implied volatility tends to the highest volatility.
- If volatility is unbounded, implied volatility must also be unbounded.
- From a trader's perspective, the more out-of-the-money (OTM) an option is, the more vol convexity it has. Provided volatility is unbounded, more OTM options must command higher implied volatility.

Asymmetric Variance Gamma Implied Variance

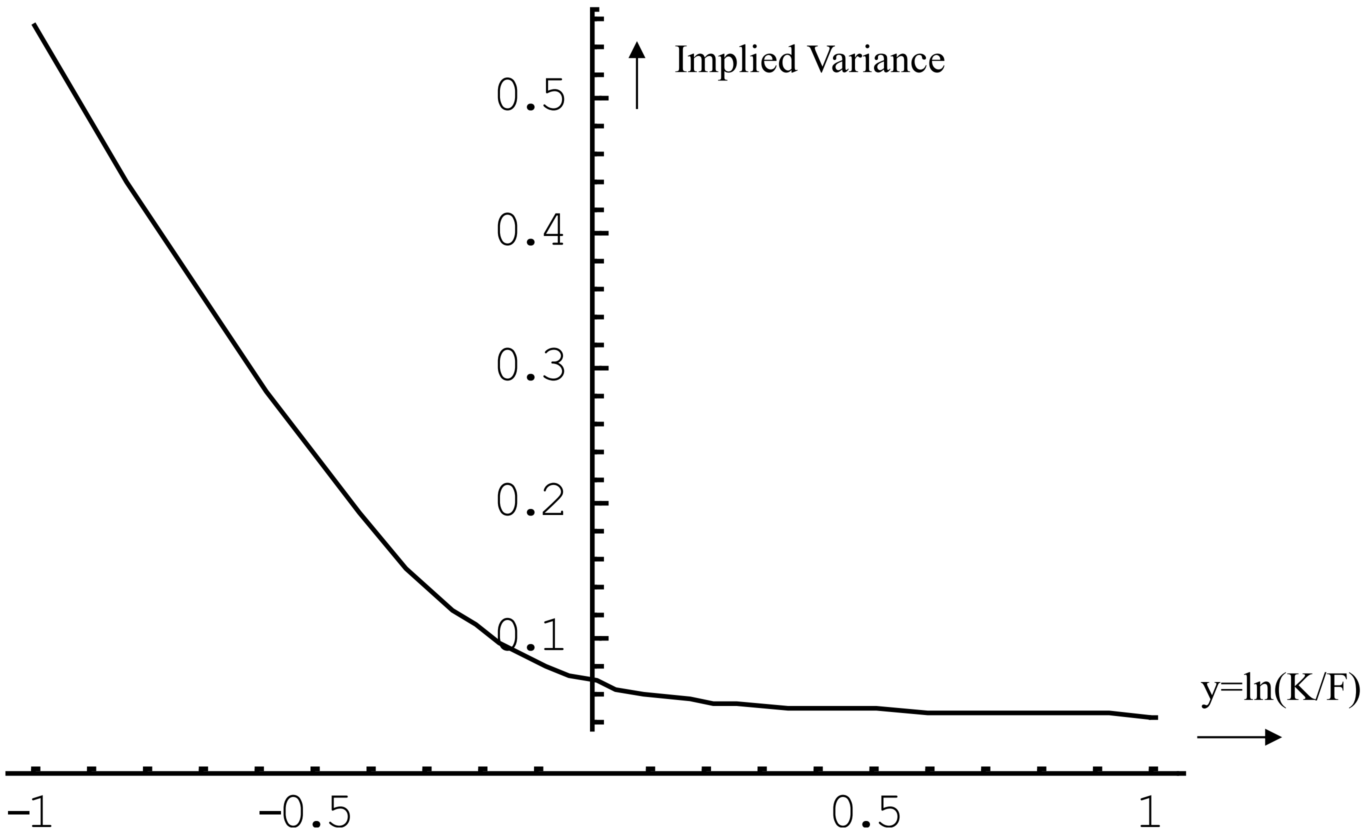


Parameters: $\overline{\mathrm{w}} = 0.04, \nu = 0.1, \theta = \text{-}1.5, \rho = -0.4$

Jump Diffusion

- Consider the simplest form of Merton's jump-diffusion model with a constant probability λ of a jump to ruin.
- Call options are valued in this model using the Black-Scholes formula with a shifted forward price.
- We graph 1 year implied variance as a function of log-strike with $v = 0.04, \lambda = 0.05$:

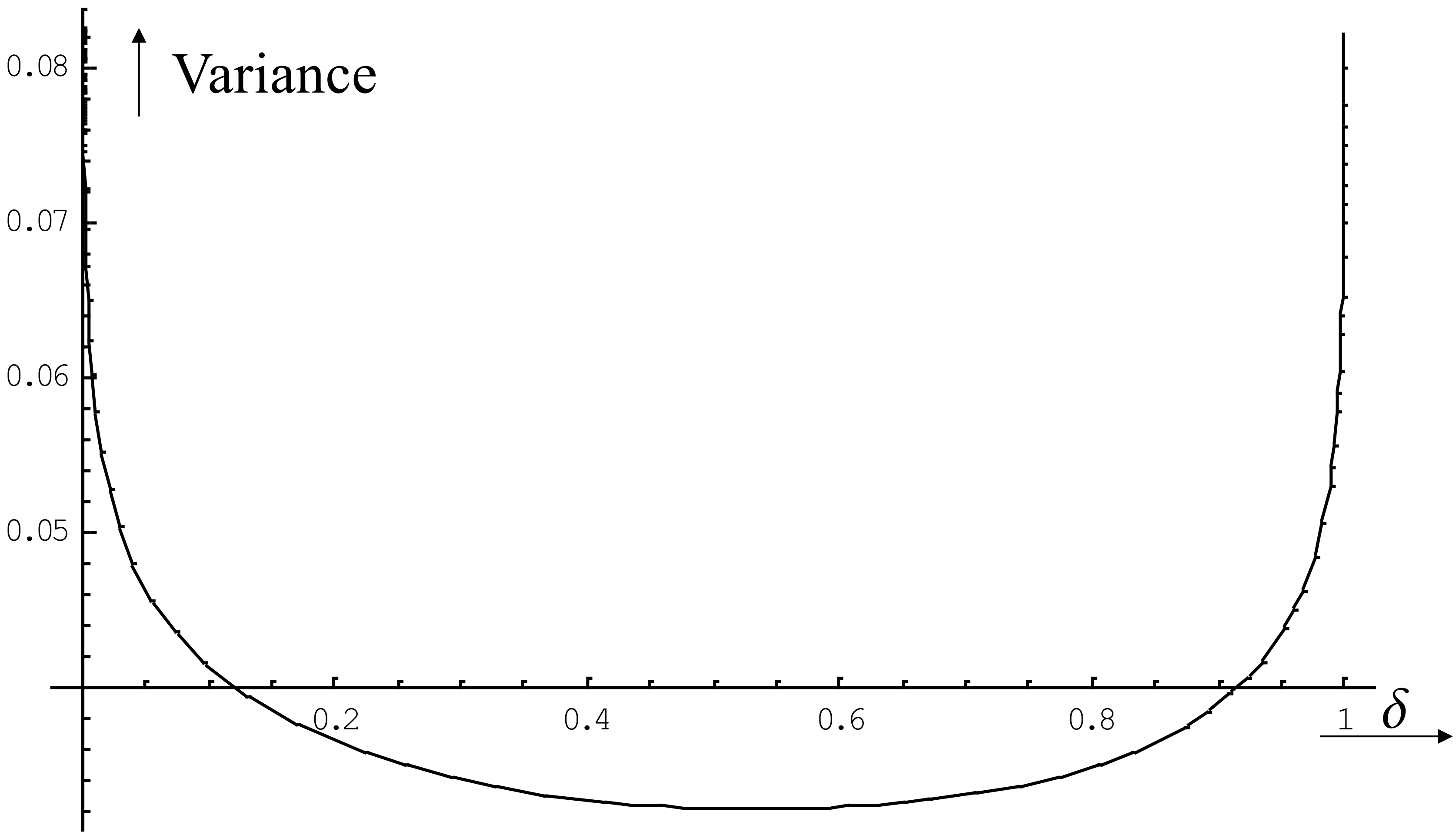
Jump-to-Ruin Model



Parameters: $\bar{v} = 0.04, \lambda = 0.05$

- So, even in jump-diffusion, v is linear in y as $|y| \to \infty$.
- In fact, we can show that for many economically reasonable stochastic-volatility-plus-jump models, implied BS variance must be asymptotically linear in the log-strike y as $|y| \to \infty$.
- This means that it does not make sense to plot implied BS variance against delta. As an example, consider the following graph of v vs. δ in the Heston model:

Variance vs δ in the Heston Model



Implications for Parameterization of the Volatility Surface

- Implied BS variance v must be parameterized in terms of the log-strike y (*vs* delta doesn't work).
- v is asymptotically linear in y as $|y| \to \infty$
- $\left.\frac{\partial v}{\partial y}\right|_{y=0}$ decays as $\frac{1}{T}$ as $T \to \infty$
- $\left.\frac{\partial v}{\partial y}\right|_{y=0}$ tends to a constant as $T \to 0$