Skew-stickiness under rough volatility

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Outline of this talk

- The skew-stickiness ratio (SSR)
- The diamond product
- Trees and forests
- The $\mathbb{F}$-expansion and stochastic volatility
    - The Bergomi-Guyon smile expansion to all orders
- The SSR in terms of diamonds for very short times
- The time series of SSR from Vola Dynamics
- Properties of the SSR

Implied volatility

According to the definition of implied volatility $\sigma_{\mathrm{BS}}(k, T)$, the market price of an option is given by

$$C(S, K, T) = C_{\mathrm{BS}}(S, K, T, \sigma_{\mathrm{BS}}(k, T))$$

where C_{BS} denotes the Black-Scholes formula and $k = \log K/S$ is the log-strike.

Hedging European options

To hedge options using the Black-Scholes formula, market makers need to hedge two effects:

- The explicit spot effect

$$\frac{\partial C}{\partial S}\,\delta S$$

and

- The change in implied volatility conditional on a change in the spot

$$\frac{\partial C}{\partial \sigma}\,\mathbb{E}\left[\delta\sigma|\delta S\right].$$

Estimating $\mathbb{E}\left[\delta\sigma(T)|\delta X\right]$

- ATM implied volatilities $\sigma(T) = \sigma_{\mathrm{BS}}(0, T)$ and stock prices are both observable.
- Market makers can estimate the second component using a simple regression:

$$\delta\sigma(T) = \beta(T)\,\frac{\delta S}{S} + \text{noise} =: \beta(T)\,\delta X + \text{noise}. \tag{1}$$

- Then

$$\beta(T) = \frac{\mathbb{E}\left[\delta\sigma(T)|\delta X\right]}{\delta X}.$$

.

The skew-stickiness ratio

- For a given time to expiration T, we define the ATM volatility skew

$$\psi(T) = \frac{\partial}{\partial k}\sigma_{\mathrm{BS}}(k,T)\bigg|_{k=0}.$$

- Bergomi [Ber09, Ber16] calls

$$\mathcal{R}(T) = \frac{\beta(T)}{\psi(T)}$$

the *skew-stickiness ratio* or *SSR*.

Sticky delta and sticky strike

In the old days, traders would typically make one of two assumptions:

- *Sticky delta* where the ATM volatility is fixed.
    - In this case, when S increases to $S + \delta S$, $\delta\sigma(T) = 0$ so $\mathcal{R}(T) = 0$.

or

- *Sticky strike* where the implied volatility is fixed for a given strike independent of the stock price.
    - In this case, when S increases to $S + \delta S$,

$$\delta\sigma(T) = \sigma_{\rm BS}(S + \delta S, T) - \sigma_{\rm BS}(S, T) \approx \psi(T)\,\delta S$$

so $\beta(T) = \psi(T)$ and $\mathcal{R} = 1$.

- Listed options were thought of as sticky strike and OTC options as sticky delta.

Empirical SSR

- Let's check some skew-stickiness ratios over the period June 1, 2010 to June 1, 2011, reproducing figures from an article I wrote with Mike Kamal [KG09] in the Encyclopedia of Quantitative Finance.

1-month SSR

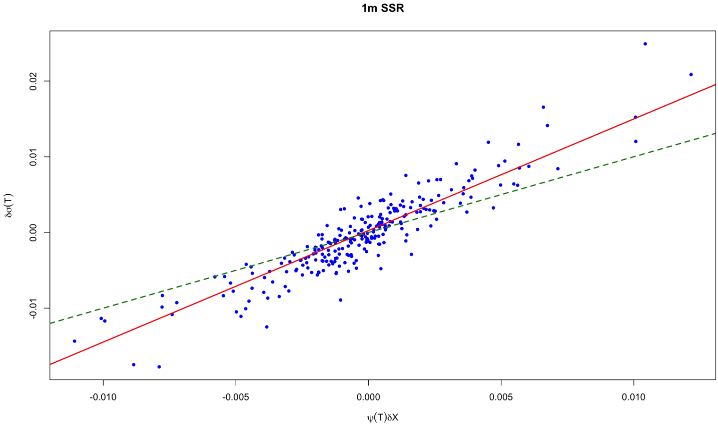


Figure 1: The 1-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

3-month SSR

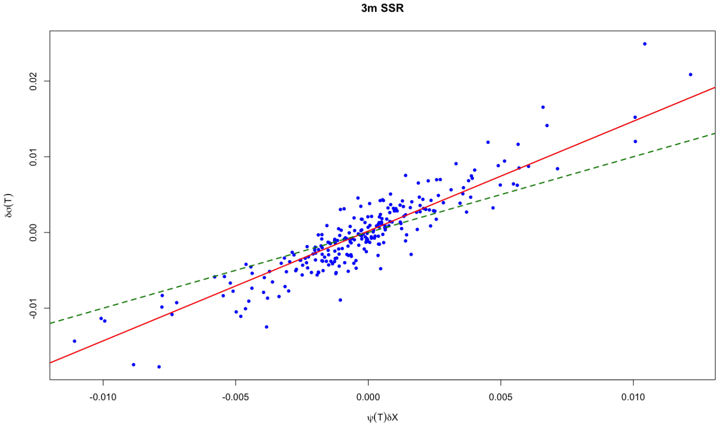


Figure 2: The 3-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

6-month SSR

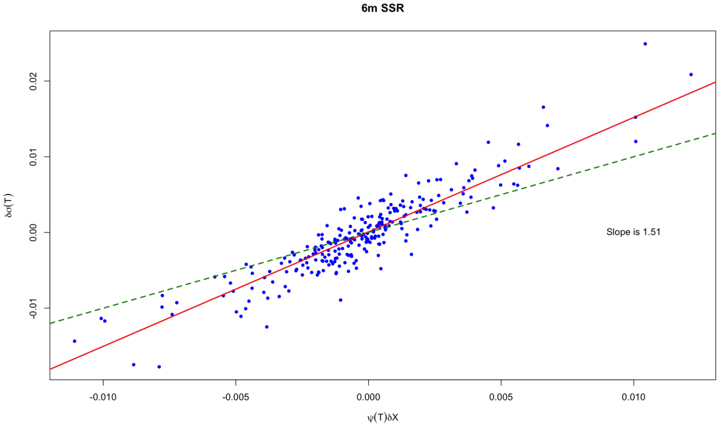


Figure 3: The 6-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

12-month SSR

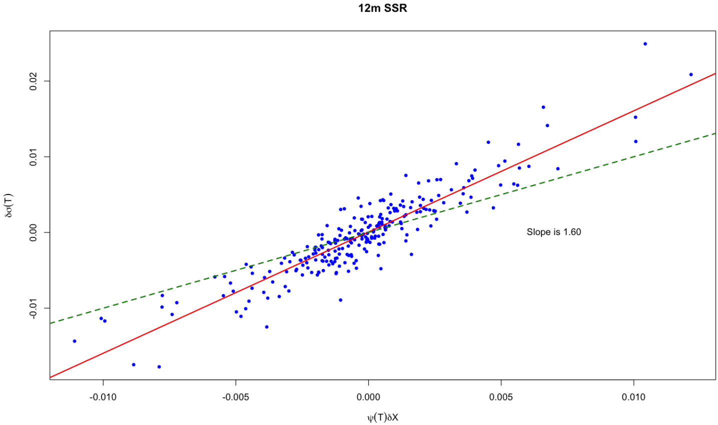


Figure 4: The 12-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

The diamond product

Definition

Given two continuous semimartingales A, B with integrable covariation process $\langle A, B\rangle$, the diamond product of A and B is another continuous semimartingale given by

$$(A \diamond B)_t(T) := \mathbb{E}\left[\langle A, B\rangle_{t,T} \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\langle A, B\rangle_T \middle| \mathcal{F}_t\right] - \langle A, B\rangle_t,$$

where $\langle A, B\rangle_{t,T} = \langle A, B\rangle_T - \langle A, B\rangle_t$.

Properties of the diamond product

- Commutative: $A \diamond B = B \diamond A$.
- Non-associative: $(A \diamond B) \diamond C \neq A \diamond (B \diamond C)$.
- $A \diamond B$ depends only on the respective martingale parts of A and B.
- $A \diamond B$ is in general not a martingale.

The $\mathbb{G}$-forest expansion

Theorem 1 (Theorem 1.1 of [FGR22])

Let Y_T be a real-valued, $\mathcal{F}_T$-measurable random variable with associated martingale $Y_t = \mathbb{E}_t[Y_T]$. Under natural integrability conditions, with a, b small enough, there is a.s. convergence of

$$\log \mathbb{E}\left[e^{aY_T + b\langle Y \rangle_T} \middle| \mathcal{F}_t \right] = aY_t + b\langle Y \rangle_t + \sum_{k \geq 2} \mathbb{G}_t^k(T), \tag{2}$$

where

$$\begin{aligned} \mathbb{G}^2 &= \left(\frac{1}{2}a^2 + b\right)(Y \diamond Y)_t(T), \\ \mathbb{G}^k &= \frac{1}{2}\sum_{j=2}^{k-2} \mathbb{G}^{k-j} \diamond \mathbb{G}^j + (a\,Y \diamond \mathbb{G}^{k-1}) \text{ for } k > 2. \end{aligned} \tag{3}$$

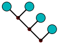
Trees and forests

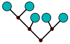
- The general term $\mathbb{G}_t^n(T)$ in (3) is naturally written as a linear combination of binary diamond trees[1].
- Hence the terminology $\mathbb{G}$-*forest expansion* for (2).
- Specifically, writing $\bullet$ as a short-hand for Y, interpreted as single leaf, we have

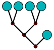
$$\mathbb{G}^2 = \left(\tfrac{1}{2}a^2 + b\right)$$

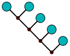

$$\mathbb{G}^3 = a\left(\tfrac{1}{2}a^2 + b\right)$$


$$\mathbb{G}^4 = \tfrac{1}{2}\left(\tfrac{1}{2}a^2 + b\right)^2$$


$$+\, a^2\left(\tfrac{1}{2}a^2 + b\right)$$


$$\mathbb{G}^5 = a\left(\tfrac{1}{2}a^2 + b\right)^2$$


$$+\, \tfrac{1}{2}a\left(\tfrac{1}{2}a^2 + b\right)^2$$


$$+\, a^3\left(\tfrac{1}{2}a^2 + b\right)$$


(4)

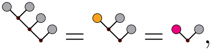
[1] Trees stolen from [Hai13]!

Forward variance models

- Let S be a strictly positive continuous martingale.
    - Then $X := \log S$ is a semimartingale with quadratic variation process $\langle X \rangle$.
- Defining $V_t\, dt := d\langle X \rangle_t$, forward variances are given by $\xi_t(u) := \mathbb{E}\left[\, V_u \middle| \mathcal{F}_t \right]$, $u > t$.
    - Forward variances are tradable assets (unlike spot variance).
    - We get a family of martingales indexed by their individual time horizons u.
- Following [BG12], it is natural to specify the dynamics of $\xi_t(u)$ for each $u > t$.
    - all conventional finite-dimensional Markovian stochastic volatility models may be cast as forward variance models.

Trees with colored leaves

- Denote $X \equiv$ .
- We could define $(X \diamond X) = M$, or  $\equiv$ , resulting in trees with leaves of two colors.
  - In a forward variance model, X_t represents the log-stock price and $M_t(T)$, the expected total variance $\int_t^T \xi_t(u)\, du$.
- In general, we can always identify subtrees in this way and assign them a new variable name (and leaf color).
  - For example, we could define  $\equiv$  to get



and so on.

The $\tilde{\mathbb{F}}$-forest expansion

A corollary of the $\mathbb{G}$-expansion is the $\tilde{\mathbb{F}}$-expansion of [AGR2020]:

Corollary

The cumulant generating function (CGF) is given by

$$\psi_t(T;a) = \log \mathbb{E}_t\left[e^{\mathrm{i}\,a\,X_T}\right] = \mathrm{i}\,a\,X_t - \frac{1}{2}a\,(a+\mathrm{i})\,M_t(T) + \sum_{\ell=1}^{\infty}\tilde{\mathbb{F}}_\ell(a). \tag{5}$$

where the $\tilde{\mathbb{F}}_\ell$ satisfy the recursion

$\tilde{\mathbb{F}}_0 = -\frac{1}{2}a(a+\mathrm{i})\,M_t = -\frac{1}{2}a(a+\mathrm{i})\bullet$ and for $k > 0$,

$$\tilde{\mathbb{F}}_\ell = \frac{1}{2}\sum_{j=0}^{\ell-2}\left(\tilde{\mathbb{F}}_{\ell-2-j}\diamond\tilde{\mathbb{F}}_j\right) + \mathrm{i}a\left(X\diamond\tilde{\mathbb{F}}_{\ell-1}\right). \tag{6}$$

Applying the recursion (6), the first few $\tilde{\mathbb{F}}$ forests are given by

$$
\begin{aligned}
\tilde{\mathbb{F}}_0 &= -\tfrac{1}{2}a(a+\mathrm{i})\,\bullet \\
\tilde{\mathbb{F}}_1 &= -\tfrac{\mathrm{i}}{2}a^2(a+\mathrm{i}) \\
\tilde{\mathbb{F}}_2 &= \tfrac{1}{2^3}a^2\,(a+\mathrm{i})^2 + \tfrac{1}{2}a^3\,(a+\mathrm{i}) \\
\tilde{\mathbb{F}}_3 &= (\tilde{\mathbb{F}}_0 \diamond \tilde{\mathbb{F}}_1) + \mathrm{i}a\,\bullet \diamond \tilde{\mathbb{F}}_2 \\
&= \tfrac{\mathrm{i}}{2^2}a^3\,(a+\mathrm{i})^2 + \tfrac{\mathrm{i}}{2^3}a^3\,(a+\mathrm{i})^2 + \tfrac{\mathrm{i}}{2}a^4\,(a+\mathrm{i}) .
\end{aligned}
$$

- Note that the total probability and martingale constraints are satisfied for each tree.
    - That is $\psi_t^T(0) = \psi_t^T(-\mathrm{i}) = 0$.

The Bergomi-Guyon smile expansion

- The Bergomi-Guyon (BG) smile expansion (Equation (14) of [BG12]) reads

$$\sigma_{\mathrm{BS}}(k, T) = \hat{\sigma}_T + \mathcal{S}_T\, k + \mathcal{C}_T\, k^2 + \mathcal{O}(\epsilon^3)$$

  where the coefficients $\hat{\sigma}_T$, $\mathcal{S}_T$ and $\mathcal{C}_T$ are complicated combinations of trees such as .
- The beauty of the BG expansion is that in some sense, it yields direct relationships between the smile and autocovariance functionals.

A formal expansion

- Regarding the forest expansion (5) as a formal power series in ϵ whose power counts the forest index ℓ, the characteristic function of the log stock price may be written in the form

$$\varphi_t(T;a) = \exp\left\{ \mathrm{i}\, a\, X_t - \frac{1}{2} a\,(a+\mathrm{i})\, M_t(T) + \sum_{\ell=1}^{\infty} \epsilon^\ell\, \tilde{\mathbb{F}}_\ell(a) \right\}.$$

- On the other hand, from for example equation (5.7) of [Gat06], with $X_t = 0$,

$$\int_0^\infty \frac{du}{u^2+\frac{1}{4}} \,\mathrm{Re}\left[e^{-\mathrm{i}uk} \left(\varphi_t^T\,(u-\mathrm{i}/2) - e^{-\frac{1}{2}\left(u^2+\frac{1}{4}\right)\Sigma(k)} \right) \right] = 0 \tag{7}$$

where, fixing T, $\Sigma(k) = \sigma_{\mathrm{BS}}^2(k,T)\,T$ is the implied total variance smile.

- Formally expand $\Sigma(k)$ as

$$\Sigma(k) = \sum_{\ell=0}^{\infty} \epsilon^{\ell}\, a_{\ell}(k).$$

  - The power of ϵ counts the order of the forest expansion.
  - We set $\epsilon = 1$ at the end of the computation.

- Equation (7) may then be rewritten in the form

$$\begin{aligned} & \int_0^{\infty} \frac{du}{u^2+\frac{1}{4}} \operatorname{Re}\left[e^{-\mathrm{i}uk} \exp\left\{-\frac{1}{2}\left(u^2+\frac{1}{4}\right) \sum_{\ell=0}^{\infty} \epsilon^{\ell}\, a_{\ell}(k)\right\}\right] \\ = \; & \int_0^{\infty} \frac{du}{u^2+\frac{1}{4}} \operatorname{Re}\left[e^{-\mathrm{i}uk}\, e^{-\frac{1}{2}\left(u^2+\frac{1}{4}\right) M_t(T)} \exp\left\{\sum_{\ell=1}^{\infty} \epsilon^{\ell}\, \tilde{\mathbb{F}}_{\ell}(u-\mathrm{i}/2)\right\}\right]. \end{aligned} \tag{8}$$

- Matching powers of ϵ on each side of (8) gives the coefficients $a_\ell(k)$ in terms of diamond trees, for any $\ell \in \mathbb{Z}^+$.

$$
\begin{aligned}
a_0(k) &= M_t(T) = \\
a_1(k) &= \left(\frac{k}{M} + \frac{1}{2}\right) \\
a_2(k) &= \frac{1}{4}(\;)^2 \left\{ -\frac{5k^2}{M^3} - \frac{2k}{M^2} + \frac{3}{M^2} + \frac{1}{4M} \right\} \\
&\quad + \frac{1}{4}(\;) \left\{ \frac{k^2}{M^2} - \frac{1}{M} - \frac{1}{4} \right\} \\
&\quad + (\;) \left\{ \frac{k^2}{M^2} + \frac{k}{M} - \frac{1}{M} + \frac{1}{4} \right\}.
\end{aligned} \tag{9}
$$

- It is straightforward to verify that the resulting expansion coincides with that of Bergomi and Guyon up to second order in ϵ.

Computation of the ATM skew $\psi(T)$

- To first order in the forest expansion,

$$\Sigma(k) \;=\; M + \epsilon \left(\frac{k}{M} + \frac{1}{2} \right) + \text{ higher order.}$$

- Wlog, set $t = 0$ and drop the subscript.
- To lowest order in the forest expansion

$$\Sigma(0) = \bullet = M(T) = \int_0^T \xi(u)\, du + \text{ higher order.}$$

and, by definition of ψ,

$$2\sqrt{M}\sqrt{T}\,\psi(T) = \Sigma'(0) = \epsilon\, \frac{1}{M} + \text{ higher order.} \qquad (10)$$

Computation of the regression coefficient $\beta(T)$

- We rewrite the regression (1) defining $\beta(T)$ as

$$\delta\sigma(0) = \frac{1}{2\sqrt{M}\sqrt{T}}\,\delta M_t(T) = \alpha + \beta(T)\,\frac{\delta S_t}{S_t} + \text{noise}.$$

- Thus, for fixed T, (reintroducing t),

$$\beta_t(T) = \frac{\mathbb{E}\left[d\langle M, X\rangle_t\right]}{2\sqrt{M}\sqrt{\tau}\,\mathbb{E}\left[d\langle X\rangle_t\right]} = \frac{-\frac{d}{dt}(X \diamond M)_t(T)}{2\sqrt{M}\sqrt{\tau}\,V_t} + \mathcal{O}(\epsilon), \tag{11}$$

where $X = \log S$, $\tau = T - t$.

  - Recall that $(X \diamond M)_t(T) = \mathbb{E}_t\left[\int_t^T d\langle X, M(T)\rangle_s\right]$.

Computation of the SSR for very short times

- Putting (10) and (11) together, we get

$$\mathcal{R}_t(T) = \frac{\beta_t(T)}{\psi_t(T)} = -\frac{M_t(T)\,\frac{d}{dt}(X \diamond M)_t(T)}{V_t\,(X \diamond M)_t(T)} + \mathcal{O}(\epsilon^2).$$

- For very short times, with $\tau = T - t$, $M_t(T) \approx V_t\,\tau$, so

$$\mathcal{R}_t(T) \approx -\tau\,\frac{d}{dt}\log(X \diamond M)_t(T).$$

To first order in the forest expansion, for τ very small:

The SSR $\mathcal{R}$ is given by the time derivative of the spot-volatility correlation functional $X \diamond M$.

A wild guess

- Suppose that something like

$$\mathcal{R}_t(T) \approx -\tau \, \frac{d}{dt} \log(X \diamond M)_t(T).$$

  holds for all expirations, as implicitly argued in [Ber16].
- Then we have the following examples:

Some SSR examples

- The SABR model
  - $(X \diamond M)_t(T) \propto (T-t)^2$ so $\mathcal{R}_t(T) \approx 2$.
- The Heston model (with $V_t = \bar{V}$),

$$(X \diamond M)_t(T) = \rho\, \eta\, \bar{V} \int_0^T dt \int_t^T e^{-\kappa\,(u-t)}\, du$$

  - For $\tau \ll 1/\kappa$, $(X \diamond M)_t(T) \sim \tau^2$ and $\mathcal{R}_t(T) \approx 2$.
  - For $\tau \gg 1/\kappa$, $(X \diamond M)(\tau) \sim \tau$ and $\mathcal{R}_t(T) \approx 1$.

The n-factor Bergomi model

- Let κ_1 be the shortest timescale (largest) mean reversion coefficient and κ_n be the longest timescale (smallest) mean reversion coefficient.
  - For $\tau \ll 1/\kappa_1$, $(X \diamond M)_t(T) \sim \tau^2$ and $\mathcal{R}_t(T) \approx 2$.
  - For $\tau \gg 1/\kappa_n$, $(X \diamond M)_t(T) \sim \tau$ and $\mathcal{R}_t(T) \approx 1$.
- For classical stochastic volatility models in general, $\mathcal{R}_t(T) \approx 2$ for τ small and $\mathcal{R}_t(T) \approx 1$ for τ large.

The skew-stickiness ratio under rough Heston

- Under rough Heston

$$(X\diamond M)_t(T) = \frac{\rho\,\nu}{\Gamma(1+\alpha)} \int_t^T \xi_t(s)\,(T-s)^\alpha\,ds \approx \frac{\rho\,\nu}{\Gamma(2+\alpha)}\,V_t\,\tau^{\alpha+1}.$$

- Then, for very short times,

$$\mathcal{R}_t(T) \approx \tau\,\frac{d}{d\tau}\log(X\diamond M)(\tau) \approx \alpha+1.$$

- Rough Heston is thus an explicit example of a model in which

$$\mathcal{R}_t(T) \approx 1+\alpha = H+\tfrac{3}{2}.$$

  - Fukasawa [Fuk21] derives the above result formally for generic rough volatility models in the limit $\tau \to 0$.

The time series of SSR from Vola Dynamics

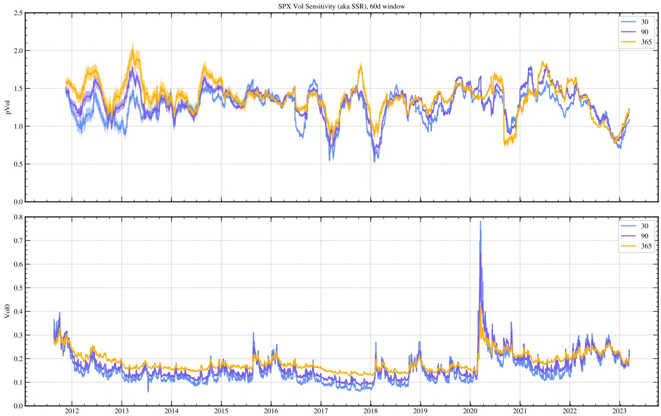


Figure 5: The 30 day, 90 day, and 1 year skew-stickiness ratios (SSR), with a trailing window of 60 days from Vola Dynamics.

The time series of SSR from Vola Dynamics

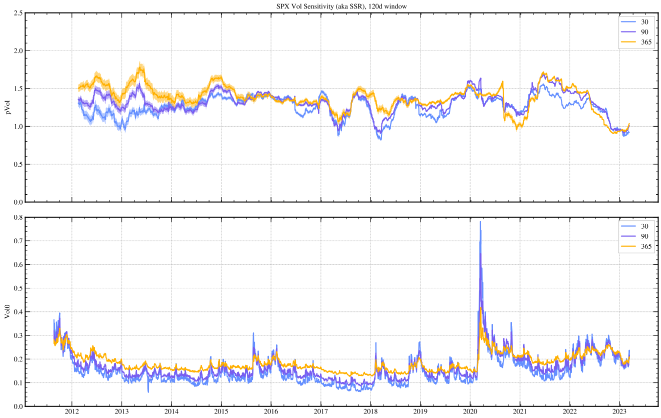


Figure 6: The 30 day, 90 day, and 1 year skew-stickiness ratios (SSR), with a trailing window of 120 days from Vola Dynamics.

Is rough volatility consistent with the SSR time series?

- To be consistent with rough volatility, we would need $\mathcal{R}(T) > \frac{3}{2}$.
  - We see that,empirically, $0.9 < \mathcal{R}(T) < 1.7$.
- Maybe volatility is super-rough?

- Moreover the previous analysis only applies in the limit $T - t = \tau \to 0$
- and, we haven't taken account of the shape of the forward variance curve.

More carefully: The SSR in AFV models

- In affine forward variance models [GKR19],

$$\frac{dS_t}{S_t} = \sqrt{V_t}\,dZ_t; \quad d\xi_t(u) = \kappa(u-t)\,\sqrt{V_t}\,dW_t.$$

- $M_t(T) = \int_t^T \xi_t(u)\,du$ so

$$dM_t(T) = \left(\int_t^T \kappa(u-t)\,du\right)\,\sqrt{V_t}\,dW_t =: \tilde{\kappa}(T-t)\,\sqrt{V_t}\,dW_t,$$

and

$$d\langle X\rangle_t = V_t\,dt; \quad d\langle X, M\rangle_t = \rho\,\tilde{\kappa}(T-t)\,V_t\,dt.$$

- Thus

$$(X \diamond M)_t(T) = \rho \int_t^T \xi_t(s)\,\tilde{\kappa}(T-s)\,ds.$$

Dependence of the SSR on ξ

- Recall that

$$\mathcal{R}_t(T) = \frac{\beta_t(T)}{\psi_t(T)} = \frac{M_t(T)\, d\langle X, M\rangle_t}{d\langle X\rangle_t\, (X \diamond M)_t(T)} + \mathcal{O}(\epsilon^2).$$

- Then, in AFV models,

$$\mathcal{R}_t(T) = \frac{\left(\int_t^T \xi_t(s)\, ds\right)\, \tilde{\kappa}(T-t)}{\int_t^T \xi_t(s)\, \tilde{\kappa}(T-s)\, ds}. \tag{12}$$

- $R_t(T)$ depends on the shape of $\xi_t(u)$!
  - A monotonic increasing ξ causes the SSR to increase, and vice versa.

Dependence of the SSR on ξ

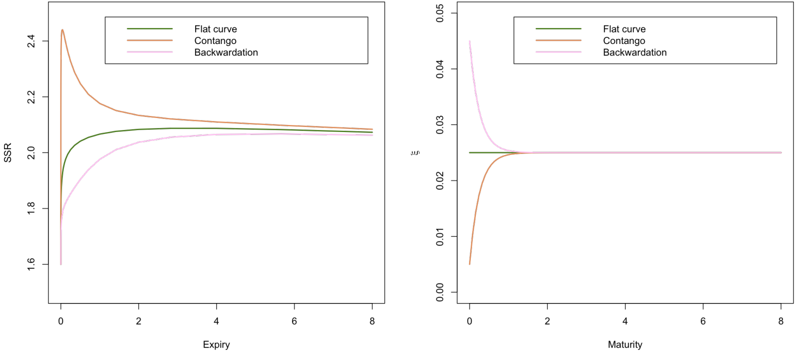


Figure 7: The SSR depends on the shape of the forward variance curve.

Path-dependence of the SSR

- With $dW_t = \rho\, dZ_t + \sqrt{1-\rho^2}\, dZ_t^{\perp}$,

$$\begin{aligned} \xi_t(u) &= \bar{\xi} + \int_{-\infty}^{t} \kappa(u-r)\, \sqrt{V_r}\, dW_r \\ &= \bar{\xi} + \rho \int_{-\infty}^{t} \kappa(u-r)\, \frac{dS_r}{S_r} + \text{independent noise.} \end{aligned}$$

- We deduce that $\mathcal{R}_t(T)$ depends on weighted average historical stock returns.
  - Historical negative returns should cause the SSR to increase, and vice versa.
  - From Figure 7, the SSR seems pretty sensitive to the shape of the forward variance curve.

How good is the skew approximation?

- Under rough Heston, with reasonable parameters:

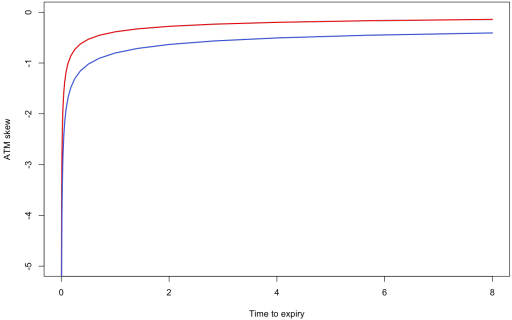


Figure 8: The rough Heston skew is in red and the first order approximation in blue.

- The first order approximation gets the shape correct but not the level.

How short is short?

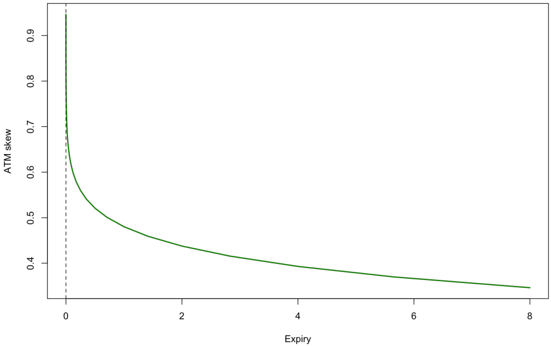


Figure 9: The ratio of the rough Heston skew to the BG first order approximation.

- The approximate formula overestimates the skew.
  - The ratio hits 0.95 for expirations later than 15 seconds!

Cancelation of errors?

- Maybe errors in the skew $\psi(T)$ are balanced to some extent by similar errors in $\beta(T)$.
- Figure 10 confirms this guess.
  - However, the skew stickiness ratio remains far above $H + \frac{3}{2}$ in the rough Heston model, for any expiration of practical interest.

- . The short term limiting expression for the SSR has no practical relevance!

SSR under rough Heston ($\lambda = 0$) for various H

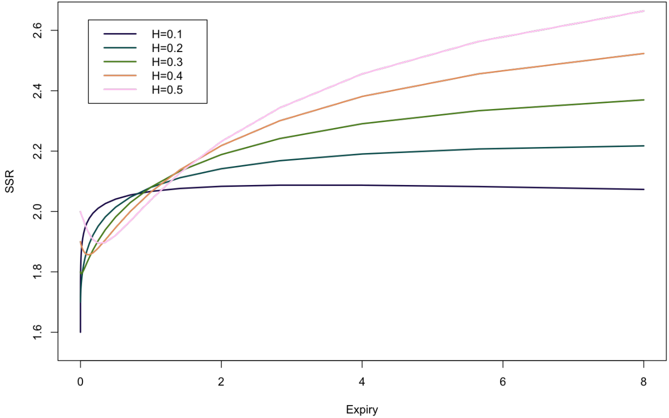


Figure 10: Theoretical SSR under rough Heston with $\xi = 0.025$ and the Padé (4,4) approximation [GR19] of the characteristic function.

Summary

- Empirically, implied volatility moves proportionally to the implied volatility skew.
  - The constant of proportionality, the SSR, is historically between 1 and 1.5.
- Under stochastic volatility, the SSR for very short expirations is given by the time derivative of $\log X \diamond M$.
  - $\mathcal{R}_t(T) \approx H + \frac{3}{2}$ where H is the Hurst exponent of V_t.
  - The SSR depends on the recent history of the underlying stock return process.
- For longer expirations, it seems that $\mathcal{R}_t(T) \gtrapprox 2$.

- The empirical behavior of the SSR seems to be inconsistent with (classical or rough) stochastic volatility!

References

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