Computing skew-stickiness

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Outline of this talk

- The skew-stickiness ratio (SSR)
- The SSR from the characteristic function
- The SSR in affine forward variance models
- Forest expansion of the SSR
  - The small time limit
  - Dependence on the forward variance curve
  - Path dependence of the SSR
- The time series of SSR from Vola Dynamics

Implied volatility

According to the definition of implied volatility $\sigma_{\text{BS}}(k, T)$, the market price of an option is given by

$$C(S, K, T) = C_{\text{BS}}(S, K, T, \sigma_{\text{BS}}(k, T))$$

where C_{BS} denotes the Black-Scholes formula and $k = \log K/S$ is the log-strike.

Updating European option prices

Market makers, when updating option prices using the Black-Scholes formula, typically consider two effects:

- The explicit spot effect

$$\frac{\partial C}{\partial S}\,\delta S$$

  and

- The change in implied volatility conditional on a change in the spot

$$\frac{\partial C}{\partial \sigma}\,\mathbb{E}\left[\delta\sigma|\delta S\right].$$

Estimating $\mathbb{E}\left[\delta\sigma(T)|\delta X\right]$

- ATM implied volatilities $\sigma_t(T) = \sigma_{\mathrm{BS},t}(0, T)$ and stock prices are both observable.
- Market makers can estimate the second component using a simple regression:

$$\delta\sigma_t(T) = \beta_t(T)\,\frac{\delta S_t}{S_t} + \text{noise} =: \beta_t(T)\,\delta X_t + \text{noise}.$$

- Then

$$\beta_t(T) = \frac{\mathbb{E}_t\left[d\left\langle \sigma(T), X\right\rangle_t\right]}{\mathbb{E}_t\left[d\left\langle X\right\rangle_t\right]}.$$

The skew-stickiness ratio

- For a given time to expiration T, we define the ATM volatility skew

$$\mathcal{S}_t(T) = \frac{\partial}{\partial k}\sigma_{\mathrm{BS}}(k, T)\Big|_{k=0}.$$

- Bergomi [Ber09, Ber16] calls

$$\mathcal{R}_t(T) = \frac{\beta_t(T)}{\mathcal{S}_t(T)}$$

the *skew-stickiness ratio* or *SSR*.

Forward variance models

- Let S be a strictly positive continuous martingale.
  - Then $X := \log S$ is a semimartingale with quadratic variation process $\langle X \rangle$.
- Defining $V_t\, dt := d\, \langle X \rangle_t$, forward variances are given by $\xi_t(u) := \mathbb{E}\left[\, V_u \middle|\, \mathcal{F}_t\right]$, $u > t$.
  - Forward variances are tradable assets (unlike spot variance).
  - We get a family of martingales indexed by their individual time horizons u.
- Following [BG12], it is natural to specify the dynamics of $\xi_t(u)$ for each $u > t$.

Our assumed model

- We would like to compute the SSR under stochastic volatility.
- Specifically, in a forward variance model of the form

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{V_t}\, dZ_t \\ d\xi_t(u) &= f_t(\xi)\, \kappa(u-t)\, dW_t, \end{aligned} \tag{1}$$

where $X = \log S$, $V_t\, dt = d\langle X \rangle_t$, and $d\langle Z, W \rangle_t = \rho\, dt$.

  - Such a model is scale-invariant, with ξ adapted to the filtration generated by W.

Implied volatility from the characteristic function

- Let $\Sigma_t(k,T) = \sigma_{BS,t}(k,T)^2\,(T-t)$.
- The Lewis representation of the option price gives Equation (5.7) of [Gat06],

$$\int_{\mathbb{R}^+} \frac{da}{a^2+\frac{1}{4}}\,\Re\left[e^{-\mathrm{i}\,a\,k}\,e^{-\left(a^2+\frac{1}{4}\right)\Sigma_t(k,T)}\right]$$
$$= \int_{\mathbb{R}^+} \frac{da}{a^2+\frac{1}{4}}\,\Re\left[e^{-\mathrm{i}\,a\,k}\varphi_t(T;a-\mathrm{i}/2)\right]. \qquad (2)$$

  - An implicit relationship between $\Sigma_t(k,T)$ and the characteristic function:

Computation of $\mathcal{S}_t(T)$

- Differentiating (2) wrt k, and performing the integration on the LHS, we obtain (5.8) of [Gat06]:

$$\mathcal{S}_t(T) = -e^{\Sigma_t(0,T)/8}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T-t}}\int_{\mathbb{R}^+}\frac{a\,da}{a^2+1/4}\Im\left[\varphi_t(T;a-\mathrm{i}/2)\right]. \tag{3}$$

  - An explicit expression for the skew.

Computation of the regression coefficient $\beta_t(T)$

- In a model of the form (1), $\sigma_t(T)$ can only depend on the forward variances $\{\xi_t(u) : u > t\}$.
- With $X = \log S$, as in Equation (9.5) of [Ber16],

$$\beta_t(T) = \frac{\mathbb{E}_t\left[d\left\langle \sigma(T), X\right\rangle_t\right]}{\mathbb{E}_t\left[d\left\langle X\right\rangle_t\right]}.$$

- Recall that by assumption,

$$\frac{dS_t}{S_t} = dX_t + \mathrm{BV} = \sqrt{\mathrm{V_t}}\,\mathrm{dZ_t},$$
$$d\xi_t(u) = f_t(\xi)\,\kappa(u-t)\,dW_t,$$

where BV denotes a bounded variation term.

- Applying Itô's Formula, denoting the Fréchet derivative by δ,

$$d\,\langle\sigma(T),X\rangle_t = \int_t^T du\,\frac{\delta\sigma_t(T)}{\delta\xi_t(u)}\;d\,\langle\xi(u),X\rangle_t$$
$$= \sqrt{V_t}\,\int_t^T du\,\frac{\delta\sigma_t(T)}{\delta\xi_t(u)}\,\rho\,f_t(\xi)\,\kappa(u-t)\,dt.$$

- This gives

$$\beta_t(T) = \frac{\rho}{\sqrt{V_t}}\,\int_t^T \frac{\delta\sigma_t(T)}{\delta\xi_t(u)}\,f_t(\xi)\,\kappa(u-t)\,du. \tag{4}$$

Nicer notation

- Let us define the operator

$$D_t^{\xi} := \frac{1}{\sqrt{V_t}} \int_t^T du\, f_t(\xi)\, \kappa(u-t)\, \frac{\delta}{\delta \xi_t(u)}.$$

- With this notation (4) becomes

$$\beta_t(T) = \rho\, D_t^{\xi} \sigma_t(T).$$

Computation of $\beta_t(T)$

- Functionally differentiating (2) with respect to $\xi_t(u)$ at $k=0$,

$$\int_{\mathbb{R}^+} da\, \Re\left[\frac{\delta \Sigma_t(0,T)}{\delta \xi_t(u)}\, e^{-\frac{1}{2}\left(a^2+\frac{1}{4}\right)\Sigma_t(0,T)}\right] = \int_{\mathbb{R}^+} \frac{da}{a^2+\frac{1}{4}}\, \Re\left[\frac{\delta}{\delta \xi_t(u)} \varphi_t(T; a - \mathrm{i}/2)\right].$$

- The LHS may be integrated explicitly to give

$$\frac{\delta \sigma_t(T)}{\delta \xi_t(u)} = e^{\frac{1}{8}\Sigma_t(0,T)} \sqrt{\frac{2}{\pi}}\, \frac{1}{\sqrt{T-t}} \int_{\mathbb{R}^+} \frac{da}{a^2+\frac{1}{4}}\, \Re\left[\frac{\delta}{\delta \xi_t(u)} \varphi_t(T; a - \mathrm{i}/2)\right].$$

- We get

$$\begin{aligned}
&\beta_t(T) \\
&= \frac{\rho}{\sqrt{V_t}} \int_t^T \frac{\delta\sigma_t(T)}{\delta\xi_t(u)}\, f_t(\xi)\, \kappa(u-t)\, du \\
&= \rho\, e^{\frac{1}{8}\,\Sigma_t(0,T)} \sqrt{\frac{2}{\pi}}\, \frac{1}{\sqrt{T-t}} \int_{\mathbb{R}^+} \frac{da}{a^2+\frac{1}{4}}\, \Re\left[D_t^{\xi}\, \varphi_t(T; a - \mathrm{i}/2)\right].
\end{aligned} \tag{5}$$

A formula for the SSR $\mathcal{R}_t(T)$

- Substituting from (3) and (5), we obtain the formal expression

$$\mathcal{R}_t(T) = -\frac{\int_{\mathbb{R}^+} \frac{da}{a^2+\frac{1}{4}}\, \rho\, \Re\left[D_t^{\xi}\varphi_t(T;a-\mathrm{i}/2)\right]}{\int_{\mathbb{R}^+} \frac{a\,da}{a^2+1/4}\, \Im\left[\varphi_t(T;a-\mathrm{i}/2)\right]}.$$

- By definition of the cumulant generating function ψ, $\varphi = e^{\psi}$ and so we get

Proposition 3.1: SSR from the characteristic function

$$\mathcal{R}_t(T) = -\frac{\int_{\mathbb{R}^+} \frac{da}{a^2+\frac{1}{4}}\, \Re\left[\rho\, D_t^{\xi}\psi_t(T;a-\mathrm{i}/2)\,\exp\left\{\psi_t(T;a-\mathrm{i}/2)\right\}\right]}{\int_{\mathbb{R}^+} \frac{a\,da}{a^2+1/4}\, \Im\left[\exp\left\{\psi_t(T;a-\mathrm{i}/2)\right\}\right]}. \tag{6}$$

Explicit computation in affine forward variance models

- In AFV models, $d\xi_t(u) = \kappa(u-t)\sqrt{V_t}\,dW_t$, so $f_t(\xi) = \sqrt{V_t}$.
- From [GKR19],

$$\psi_t(T;a) = \log\varphi_t(T;a) = \int_t^T \xi_t(s)\,g(T-s;a)\,ds, \tag{7}$$

where g satisfies the convolution Riccati equation

$$g(\tau;a) = -\frac{1}{2}\,a(a+\mathrm{i}) + \mathrm{i}\,\rho\,a\,(\kappa\star g)(\tau;a) + \frac{1}{2}\,(\kappa\star g)(\tau;a)^2. \tag{8}$$

- Functionally differentiating (7) gives, for $u \in [t, T]$,

$$\frac{\delta}{\delta\xi_t(u)}\psi_t(T;a) = g(T-u;a).$$

- Thus,

$$D_t^\xi \psi_t(T; a - \mathrm{i}/2) = (\kappa \star g)(T - t; a - \mathrm{i}/2).$$

- Then, from (6), we obtain

SSR in affine forward variance models

$$\mathcal{R}_t(T) = -\frac{\int_{\mathbb{R}^+} \frac{da}{a^2 + \frac{1}{4}} \Re \left[\rho\, (\kappa \star g)(T - t; a - \mathrm{i}/2)\, e^{\int_t^T \xi_t(s)\, g(T-s; a - \mathrm{i}/2)\, ds} \right]}{\int_{\mathbb{R}^+} \frac{a\, da}{a^2 + 1/4} \Im \left[e^{\int_t^T \xi_t(s)\, g(T-s; a - \mathrm{i}/2)\, ds} \right]}. \tag{9}$$

- Given a solution $g(\cdot)$ of the convolution Riccati equation (8), we may evaluate (9) numerically.
    - In particular, $\mathcal{R}_t(T)$ may be evaluated in the rough Heston model.

SSR under rough Heston ($\lambda = 0$) for various H

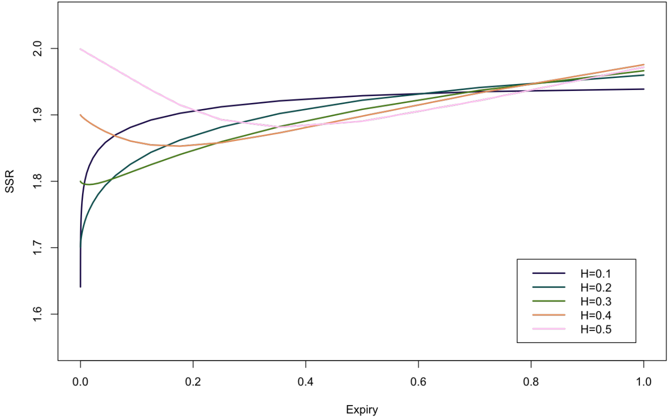

Figure 1: The rough Heston SSR with $\xi = 0.025$, $\rho = -0.8$, $\nu = 0.4$ and the Padé (5,5) approximation [GR19, GR24] of the rough Heston solution.

The diamond product

Definition

Given two continuous semimartingales A, B with integrable covariation process $\langle A, B \rangle$, the diamond product of A and B is another continuous semimartingale given by

$$(A \diamond B)_t(T) := \mathbb{E}\left[\langle A, B \rangle_{t,T} \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\langle A, B \rangle_T \middle| \mathcal{F}_t\right] - \langle A, B \rangle_t,$$

where $\langle A, B \rangle_{t,T} = \langle A, B \rangle_T - \langle A, B \rangle_t$.

Diamond trees and forests

- The diamond product of two trees $\mathbb{T}_1$ and $\mathbb{T}_2$ is represented by *root joining*,

$$\mathbb{T}_1 \diamond \mathbb{T}_2 = $$  .

  - The two binary trees $\mathbb{T}_1$ and $\mathbb{T}_2$ are represented as the single leaves ● and ●.
- We regard linear combinations of diamond trees as *forests*.
- In what follows,
  - ● $= X_t$,
  - ● $= M_t(T) = (X \diamond X)_t(T) = \int_t^T \xi_t(u)\, du$.

The $\tilde{\mathbb{F}}$-forest expansion

With $M_t(T) = \int_t^T \xi_t(u)\, du$, the $\tilde{\mathbb{F}}$-expansion of [AGR2020] reads:

The forest expansion

The cumulant generating function (CGF) is given by

$$\psi_t(T;a) = \log \mathbb{E}_t \left[e^{\mathrm{i}\, a\, X_T} \right] = \mathrm{i}\, a\, X_t - \frac{1}{2} a\,(a + \mathrm{i})\, M_t(T) + \sum_{\ell=1}^{\infty} \tilde{\mathbb{F}}_\ell(a). \tag{10}$$

where the $\tilde{\mathbb{F}}_\ell$ satisfy the recursion

$\tilde{\mathbb{F}}_0 = -\frac{1}{2} a(a+\mathrm{i})\, M_t = -\frac{1}{2} a(a+\mathrm{i}) \bullet$ and for $k > 0$,

$$\tilde{\mathbb{F}}_\ell = \frac{1}{2} \sum_{j=0}^{\ell-2} \left(\tilde{\mathbb{F}}_{\ell-2-j} \diamond \tilde{\mathbb{F}}_j \right) + \mathrm{i}a \left(X \diamond \tilde{\mathbb{F}}_{\ell-1} \right).$$

Second order computation of $\mathcal{R}_t(T)$

- Consider a formal expansion according to values of ℓ.
- From (10), to second order in the forest expansion,

$$\psi_t(T; a - \mathrm{i}/2)$$

$$= -\frac{1}{2}\left(a^2 + \frac{1}{4}\right)\Big\{$$  $$+ \left(\mathrm{i}\, a + \frac{1}{2}\right)$$  $$- \frac{1}{4}\left(a^2 + \frac{1}{4}\right)$$  $$- (a - \mathrm{i}/2)^2$$  $$\Big\}$$

  - The forest expansion (10) is effectively a small ν (vol-of-vol) expansion where for fixed a, the $\tilde{\mathbb{F}}_\ell(\tau)$ scale as $\tau^{\ell\,\alpha+1}$ as $\tau \downarrow 0$.
  - When computing $\mathcal{R}_t(T)$ using (6), different powers of a will, after integration, generate different powers of τ.
  - In our paper, we take powers of a into account to get a next-to-leading-order small τ expansion.
- Let's continue with our expansion to second order in ν...

- First we simplify the numerator of (6):

$$\Re\left[e^{\psi_t(T;a-\mathrm{i}/2)}\,D_t^{\xi}\left\{\bullet+(\mathrm{i}\,a+\tfrac{1}{2})\right\}\right]$$
$$\approx e^{-\frac{1}{2}\left(a^2+\frac{1}{4}\right)\,M}\,D_t^{\xi}\left\{\bullet+\tfrac{1}{2}\right\}.$$

- Then we simplify the denominator:

$$\Im\left[e^{\psi_t(T;a-\mathrm{i}/2)}\right]=-\tfrac{1}{2}\,e^{-\frac{1}{2}\left(a^2+\frac{1}{4}\right)\,M}\,a\,\left(a^2+\tfrac{1}{4}\right)\left\{\;+\;\right\}.$$

- This gives

Second order forest expansion of SSR

$$\mathcal{R}_t(T)\approx\frac{M_t(T)\,\rho\,D_t^{\xi}\left\{\bullet+\tfrac{1}{2}\right\}}{\;+\;}.$$

Computation of $\mathcal{R}_t(T)$ to leading order

- Writing out D^{ξ} and  explicitly gives:

Lemma 4.1

To leading order,

$$\mathcal{R}_t(T) = \frac{M_t(T)}{\sqrt{V_t}} \, \frac{\int_t^T \, f_t(\xi)\, \kappa(u-t)\, du}{\int_t^T \, ds \, \int_s^T \, \mathbb{E}_t \left[\sqrt{V_s}\, f_s(\xi)\right] \, \kappa(u-s)\, du}. \tag{11}$$

The small time limit

- In the limit $T \to t$,we obtain

Corollary

Let $\tau = T - t$. Then

$$\lim_{T \to t} \mathcal{R}_t(T) = \tau \, \frac{d}{d\tau} \log \left(\int_0^\tau ds \int_0^s \kappa(u)\, du \right).$$

- The following corollary confirms a formal computation of Fukasawa in Remark 2.10 of [Fuk21].

Corollary

Let $\kappa(s) = s^{\alpha - 1}\, L_\kappa(s)$ where L_κ is a slowly varying function. Then

$$\lim_{T \to t} \mathcal{R}_t(T) = \alpha + 1.$$

Dependence of $\mathcal{R}_t(T)$ on $\xi_t(u)$

- We rewrite the leading order expression (11) suggestively in the form

$$\mathcal{R}_t(T) = \frac{\left(\int_t^T \xi_t(s)\, ds\right) \int_t^T \sqrt{V_t}\, f_t(\xi)\, \kappa(u-t)\, du}{V_t \int_t^T ds \int_s^T \mathbb{E}_t\left[\sqrt{V_s}\, f_s(\xi)\right] \kappa(u-s)\, du}.$$

- We observe that this expression should be rather insensitive to the level of the forward variance curve.
- However, $\mathcal{R}_t(T)$ is sensitive to the shape of $\xi_t(u)$.
  - A monotonic increasing forward variance curve will cause the SSR to increase relative to the flat curve case, and vice versa.

The affine case

- In the case of AFV models,

$$\mathcal{R}_t(T) = \frac{\left(\int_t^T \xi_t(s)\,ds\right)\,\tilde{\kappa}(T-t)}{\int_t^T \xi_t(s)\,\tilde{\kappa}(T-s)\,ds},$$

  where $\tilde{\kappa}(\tau) = \int_0^\tau \kappa(s)\,ds$.
- If the forward variance curve is flat with $\xi_t(u) = \bar{V}$, the SSR does not depend on the level $\bar{V}$ at all!
- However, once again, $\mathcal{R}_t(T)$ does depend on the shape of $\xi_t(u)$.

An example: Rough Heston

- In Figure 2, with parameters $H = 0.1$; $\nu = 0.4$; $\rho = -.8$ we plot the rough Heston SSR using the AFV formula (9).
- We assuming three different forward variance curves of the form:

$$\xi_t(u) = (V_t - \bar{V})\, e^{-\lambda t} + \bar{V},$$

with $\bar{V} = 0.025$, $\lambda = 7$ and $V_t = 0.025$ (flat), $V_t = 0.005$ (contango), and $V_t = 0.045$ (backwardation).
- We observe that the SSR is very sensitive to the shape of the forward variance curve.

An example: Rough Heston

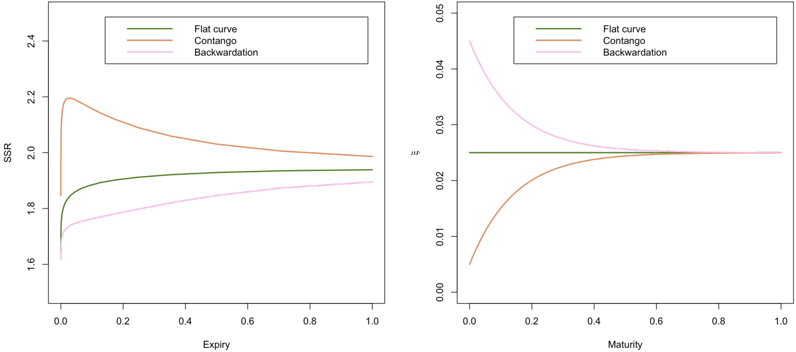

Figure 2: The rough Heston SSR with various forward variance curves, and parameters $H = 0.1$; $\nu = 0.4$; $\rho = -.8$. The left-hand plot is of the SSRs and the right-hand plot shows the assumed forward variance curves.

Path-dependence of the SSR

- In an AFV model, with $dW_t = \rho\, dZ_t + \sqrt{1-\rho^2}\, dZ_t^{\perp}$,

$$\begin{aligned}\xi_t(u) &= \bar{\xi} + \int_{-\infty}^{t} \kappa(u-r)\,\sqrt{V_r}\,dW_r \\ &= \bar{\xi} + \rho \int_{-\infty}^{t} \kappa(u-r)\,\frac{dS_r}{S_r} + \text{independent noise.}\end{aligned}$$

- The forward variance curve depends on a weighted average of historical stock returns.
- Thus $\mathcal{R}_t(T)$ also depends on weighted average historical stock returns.
    - If recent returns are very negative, we expect $\xi_t(u)$ to be backwardated, lowering $\mathcal{R}_t(T)$, and vice versa.
- This argument goes through for every forward variance model.
    - The forward variance curve is a noisy transform of the historical series of stock returns.

Dependence of the SSR on the kernel

- We choose three rough Heston parameter sets that generate approximately the same 1m, 3m ,6m and 12m smiles.
  - $\lambda = 0,\ 1,\ 2$ respectively.

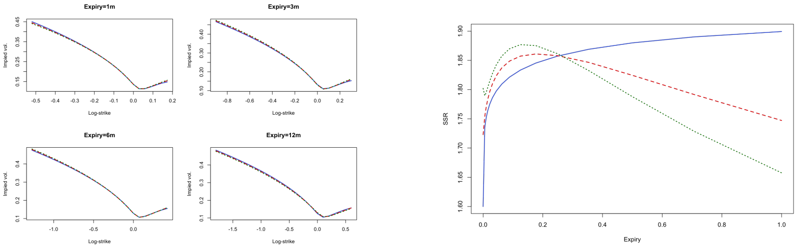

Figure 3: Left: 1m, 3m ,6m and 12m smiles. Right: Corresponding SSR plots.

- This shows that the SSR cannot be deduced in a model-free way from the volatility surface.
  - The SSR is sensitive to precise dynamical assumptions.

Working paper

- For more details and other computations, please see [FG24].

The time series of SSR from Vola Dynamics

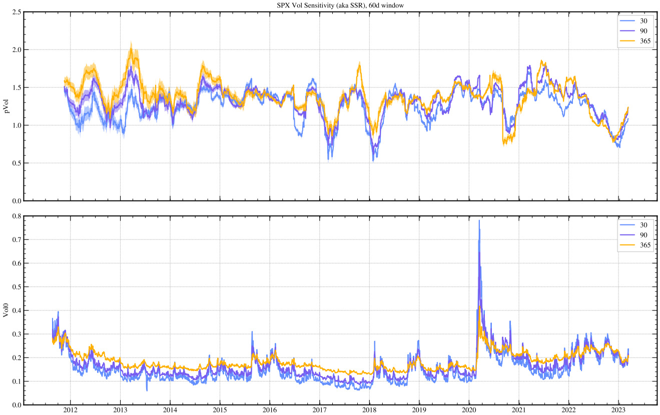

Figure 4: The 30 day, 90 day, and 1 year skew-stickiness ratios (SSR), with a trailing window of 60 days from Vola Dynamics.

Is stochastic volatility consistent with the SSR time series?

- To be consistent with rough volatility, we would need $\mathcal{R}(T) > \frac{3}{2}$.
  - We see that,empirically, $0.9 < \mathcal{R}(T) < 1.7$.
- It certainly seems that the empirically observed SSR is inconsistent with any affine stochastic volatility model.
  - And from [BDDM24], also inconsistent with rough Bergomi.
- Is the empirically observed SSR is consistent with any stochastic volatility model?

- Figure 3 demonstrates that the SSR is highly dependent on assumed model dynamics.
  - This gives us hope that a model may be found that generates SSRs consistent with observation.
- Indeed, very recent results of Shaun Li in his thesis [Li24] suggest that the quintic model can jointly fit SPX and VIX, and generate reasonable values of the SSR!
  - Underlying the analysis is a signature expansion of the characteristic function [AJG24] and a version of our Proposition 3.1.
  - It seems that Equation (6) may be applicable to a much wider class of models than we originally thought.

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