Valuation of Volatility Derivatives

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Outline of this talk

- Valuing variance swaps under compound Poisson assumptions
- The impact (or lack thereof) of jumps on the valuation of variance swaps
- Finding the risk neutral distribution of quadratic variation
- Options on quadratic variation
- How to value VIX futures
- Estimating volatility of volatility
- Volatility of volatility as a traded parameter

Quadratic variation for a compound Poisson process

- Let X_T denote the return of a compound Poisson process so that

$$X_T = \sum_i^{N_T} Y_i$$

  with Y_i iid and N_T a Poisson process with mean λT.

- Define the quadratic variation as

$$\langle X \rangle_T = \sum_i^{N_T} |Y_i|^2$$

$$\mathbb{E}\left[\langle X \rangle_T\right] = \mathbb{E}[N_T]\mathbb{E}\left[|Y_i|^2\right] = \lambda T \int y^2 \, \mu(y)\, dy$$

- Also,

$$\mathbb{E}[X_T] = \lambda T \int y \, \mu(y)\, dy$$

$$\mathbb{E}\left[X_T^2\right] = \lambda T \int y^2 \, \mu(y)\, dy + (\lambda T)^2 \left(\int y \, \mu(y)\, dy\right)^2$$

- So $\mathbb{E}\left[\langle X \rangle_T\right] = \mathrm{var}[X_T]$ Expected QV = Variance of terminal distribution for compound Poisson processes! Obviously not true in general (*e.g.* Heston).

Examples of compound Poisson processes

- Merton jump-diffusion model (constant volatility lognormal plus independent jumps).

- AVG (Asymmetric variance gamma)

- CGMY (More complicated version of AVG)

- NIG (Normal inverse Gaussian)

- List does not include time-changed models such as VG-CIR

Valuing variance swaps

- We can express the first two moments of the final distribution in terms of strips of European options as follows:

$$\mathbb{E}[X_T] = \mathbb{E}[\ln(S_T / S_0)] = -\int_{-\infty}^{0} dk \, p(k) - \int_{0}^{\infty} dk \, c(k)$$

$$\mathbb{E}[X_T^2] = \mathbb{E}[\ln(S_T / S_0)^2] = -\int_{-\infty}^{0} dk \, 2k \, p(k) - \int_{0}^{\infty} dk \, 2k \, c(k)$$

- So, if we know European option prices, we may compute expected quadratic variation – *i.e.* we may value variance swaps as

$$\mathbb{E}[\langle X \rangle_T] = \text{var}[X_T] = \mathbb{E}[\langle X \rangle_T^2] - \mathbb{E}[\langle X \rangle_T]^2$$

Valuing variance swaps under diffusion assumptions

- If the underlying process is a diffusion, expected quadratic variation may be expressed in terms of an infinite strip of European options (the log-strip):

$$\mathbb{E}\left[\langle X \rangle_T\right] = \int_{-\infty}^{0} dk\, 2\, p(k) + \int_{0}^{\infty} dk\, 2\, c(k)$$

- Equivalently, we can compute expected quadratic variation directly from implied volatilities without computing intermediate option prices using the formula

$$\mathbb{E}\left[\langle X \rangle_T\right] = \int_{-\infty}^{\infty} dz\, N'(z)\, \sigma_{BS}^{2}(z)$$

with

$$z \equiv d_2(k) = \frac{-k}{\sigma_{BS}(k,T)\sqrt{T}} - \frac{\sigma_{BS}(k,T)\sqrt{T}}{2}$$

What is the impact of jumps?

- In summary, if the underlying process is compound Poisson, we have the above formula to value a variance swap in terms of a strip of European options and if the underlying process is a diffusion, we have the usual well-known formula.

- In reality, we don't know the underlying process but we do know the prices of European options.
- Suppose we were to assume a diffusion but the underlying process really had jumps. What would the practical valuation impact be?

The jump correction to variance swap valuation

- Once again, if the underlying process is a diffusion, we can value a variance swap in terms of the log-strip:

$$\mathbb{E}\left[\langle X \rangle_T\right] = -2\mathbb{E}\left[X_T\right]$$

- Also,

$$\mathbb{E}\left[X_T\right] = -i\,\partial_u\mathbb{E}\left[e^{iuX_T}\right]\Big|_{u=0} = -i\,\partial_u\phi_T(u)\Big|_{u=0}$$

where $\phi_T(u)$ is the characteristic function.

- Our assumptions that jumps are independent of the diffusion leads to factorization of the characteristic function into a diffusion piece and a pure jump piece.

$$\phi_T(u) = \phi_T^C(u)\,\phi_T^J(u)$$

- From the Lévy-Khintchine representation, we arrive at

$$-i\,\partial_u\phi_T^J(u)\Big|_{u=0} = \lambda T\left[\int(1+y-e^y)\,\mu(y)\,dy\right]$$

The jump correction continued

- On the other hand, we already showed that

$$\mathbb{E}\left[\left\langle X^J \right\rangle_T\right] = \mathrm{var}\left[X_T^J\right] = \lambda T\left[\int y^2 \mu(y)\,dy\right]$$

where the superscript J refers to the jump component of the process.

- It follows that the difference between the fair value of a variance swap and the value of the log-strip is given by

$$\mathbb{E}\left[\left\langle X \right\rangle_T\right] + 2\mathbb{E}\left[\ln(S_T/S_0)\right] = 2\lambda T\left[\int (1 + y + y^2/2 - e^y)\,\mu(y)\,dy\right]$$

- Example:
  - Lognormally distributed jumps with mean α and standard deviation δ

$$\mathbb{E}\left[\left\langle X \right\rangle_T\right] + 2\mathbb{E}\left[\ln(S_T/S_0)\right] = 2\lambda T\left[1 + \alpha + \frac{\alpha^2 + \delta^2}{2} - e^{\alpha^2 + \delta^2/2}\right]$$

$$= -\frac{\lambda T}{3}\alpha\left(\alpha^2 + 3\delta^2\right) + O(\alpha^4)$$

  - With $\alpha = -0.09, \delta = 0.14$ and $\lambda = 0.61$ (from BCC) we get a correction of 0.00122427 per year which at 20% vol. corresponds to 0.30% in volatility terms.

Remarks

- Jumps have to be extreme to make any practical difference to the valuation of variance swaps.

- The standard diffusion-style valuation of variance swaps using the log-strip works well in practice for indices

  - From the perspective of the dealer hedging a variance swap using the log-strip, the statistical measure is the relevant one.

    - How often do jumps occur in practice and how big are they?

  - Jumps in the risk-neutral measure are driven by the short-dated smile. However, the model may be mis-specified. Jumps may not be the main reason that the short-dated skew is so steep.

- Single stocks may be another story – jumps tend to be frequent even in the statistical measure.

A simple lognormal model

- Define the quadratic variation

$$\langle X \rangle_T := \int_0^T \sigma^2(s, \omega)\, ds$$

- Assume that $\log\left(\sqrt{\langle X \rangle_T}\right)$ is normally distributed with mean μ and variance s^2.

- Then $\log\left(\langle X \rangle_T\right)$ is also normally distributed with mean 2μ and variance $4s^2$.

- Volatility and variance swap values are given by respectively

$$\mathbb{E}\left[\sqrt{\langle X \rangle_T}\right] = e^{\mu + s^2/2}; \mathbb{E}\left[\langle X \rangle_T\right] = e^{2\mu + 2s^2}$$

- Solving for μ and s^2 gives

$$s^2 = 2\log\left(\frac{\sqrt{\mathbb{E}\left[\langle X \rangle_T\right]}}{\mathbb{E}\left[\sqrt{\langle X \rangle_T}\right]}\right); \ \mu = \log\left(\frac{\mathbb{E}\left[\sqrt{\langle X \rangle_T}\right]^2}{\sqrt{\mathbb{E}\left[\langle X \rangle_T\right]}}\right)$$
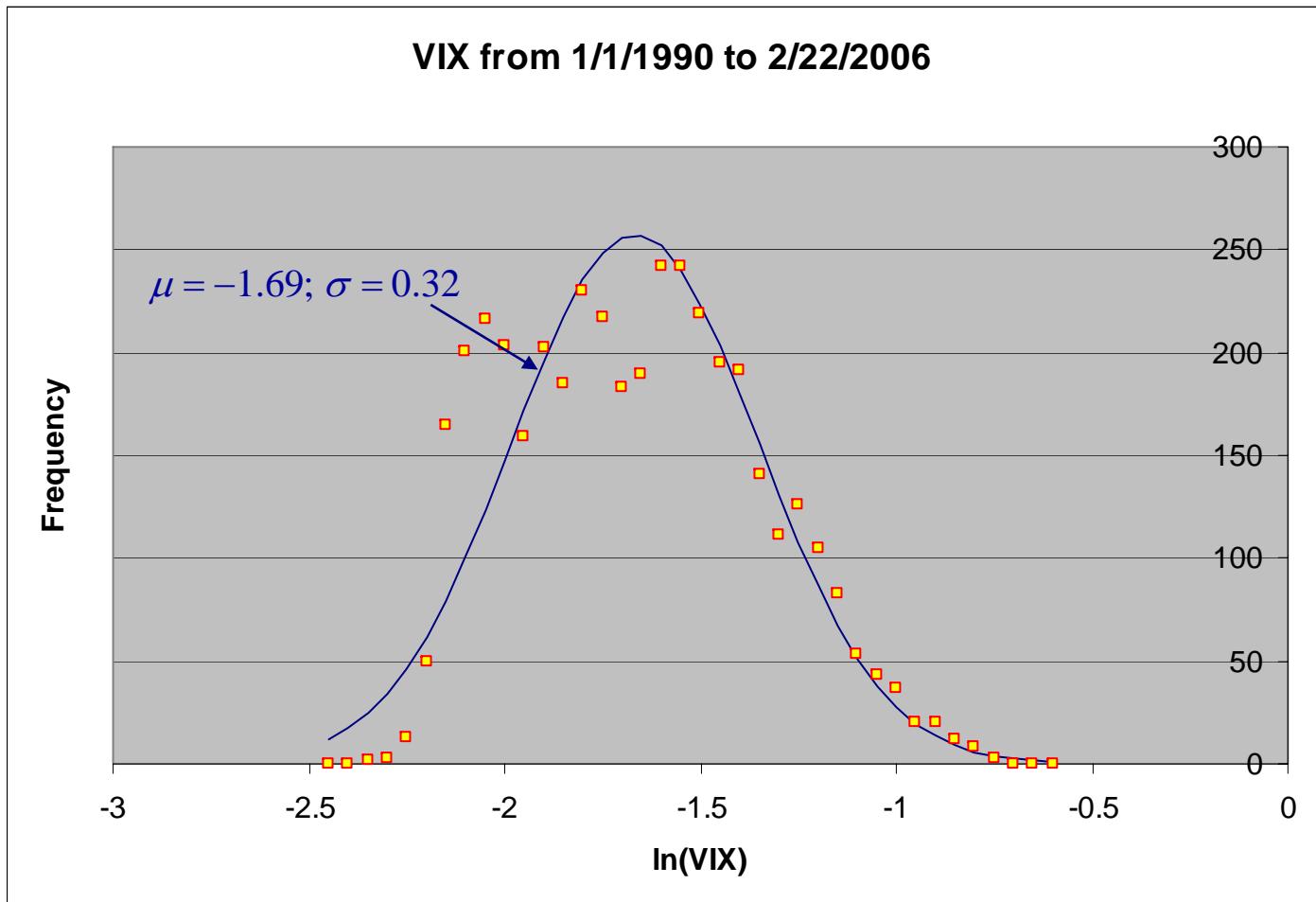
A simple lognormal model continued

- Note that under this lognormal assumption, the convexity adjustment (between volatility and variance swaps) is given by

$$\sqrt{\mathbb{E}\left[\langle X \rangle_T\right]} - \mathbb{E}\left[\sqrt{\langle X \rangle_T}\right] = \left(e^{s^2/2} - 1\right)\mathbb{E}\left[\sqrt{\langle X \rangle_T}\right]$$

- In the lognormal model, given volatility and variance swap prices, the entire distribution is specified and we may price any claim on quadratic variation!

- Moreover, the lognormal assumption is reasonable and widely assumed by practitioners.

Unconditional distribution of VIX vs lognormal

VIX from 1/1/1990 to 2/22/2006

$\mu = -1.69; \ \sigma = 0.32$

Frequency

ln(VIX)

Consistent with lognormal volatility dynamics!

Vol term structure and skew under stochastic volatility

- All stochastic volatility models generate volatility surfaces with approximately the same shape

- The Heston model $dv = -\lambda\left(v - \bar{v}\right)dt + \eta\sqrt{v}\, dZ$ has an implied volatility term structure that looks to leading order like

$$\sigma_{BS}\left(x,T\right)^2 \approx \bar{v} + \left(v - \bar{v}\right)\frac{\left(1 - e^{-\lambda T}\right)}{\lambda T}$$

  It's easy to see that this shape should not depend very much on the particular choice of model.
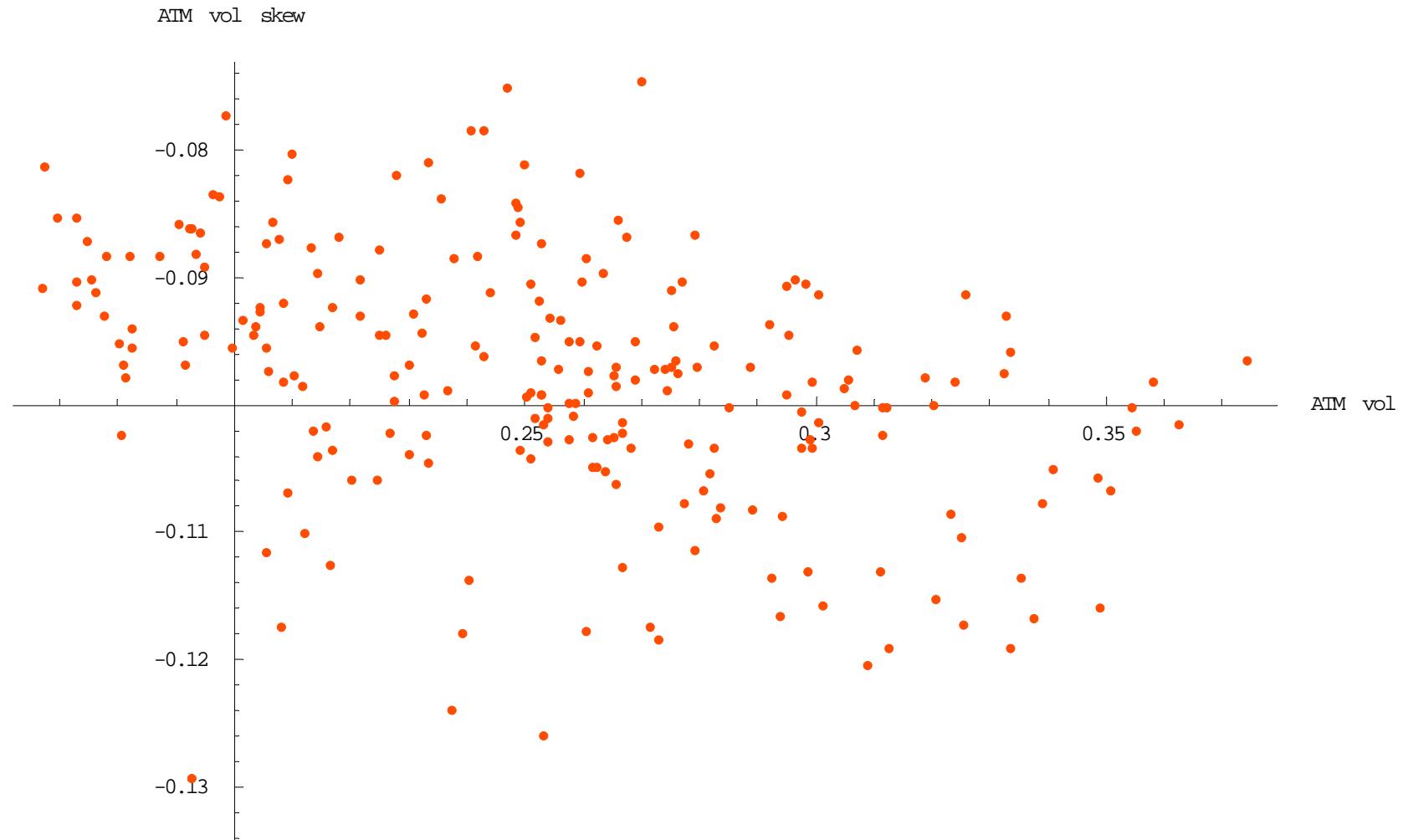
- Also, Gatheral (2004) shows that the term structure of the at-the-money volatility skew has the following approximate behavior for all stochastic volatility models of the form $dv = -\lambda\left(v - \bar{v}\right)dt + \eta v^{\beta}\, dZ$

$$\left.\frac{\partial}{\partial x}\sigma_{BS}\left(x,T\right)^2\right|_{x=0} \approx \frac{\rho\eta\, v^{\beta-1/2}}{\lambda'T}\left\{1 - \frac{\left(1 - e^{-\lambda'T}\right)}{\lambda'T}\right\}$$

  with $\lambda' = \lambda - \rho\eta\, v^{\beta-1/2}/2$

- So we can estimate β by regressing volatility skew against volatility level.

SPX 3-month ATM volatility skew vs ATM 3m volatility

Interpreting the regression of skew vs volatility

- Recall that if the variance satisfies the SDE

$$dv \sim v^{\beta} \, dZ$$

at-the-money variance skew should satisfy

$$\left. \frac{\partial v}{\partial k} \right|_{k=0} \propto v^{\beta - 1/2}$$

and at-the-money volatility skew should satisfy

$$\left. \frac{\partial \sigma_{BS}}{\partial k} \right|_{k=0} \propto \sigma_{BS}^{\,2\beta - 2}$$

- The graph shows volatility skew to be roughly independent of volatility level so $\beta \approx 1$ again consistent with lognormal volatility dynamics.

A variance call option formula

■ Assuming this lognormal model, we obtain a Black-Scholes style formula for calls on variance:

$$\mathbb{E}\left[\langle X \rangle_T - K\right]^+ = e^{2\mu + 2s^2} N(d_1) - K N(d_2)$$

with
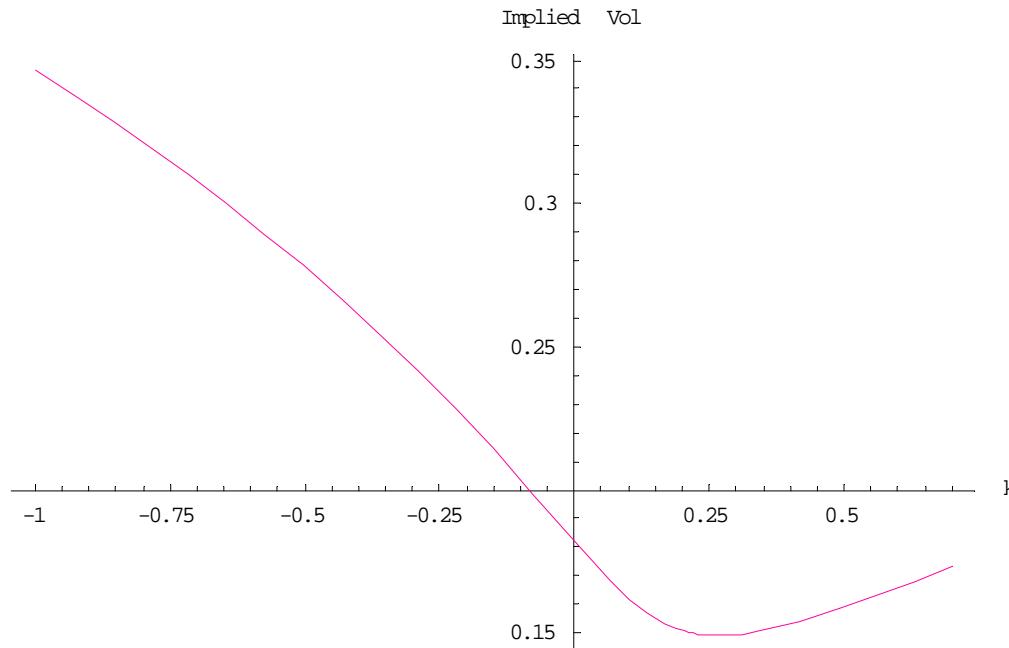
$$d_1 = \frac{-\frac{1}{2}\log K + \mu + 2s^2}{s}; d_2 = \frac{-\frac{1}{2}\log K + \mu}{s}$$

Example: One year Heston with BCC parameters

- We compute one year European option prices in the Heston model using parameters from Bakshi, Cao and Chen. Specifically

$$v = \overline{v} = 0.04; \eta = 0.39; \lambda = 1.15; \rho = -0.64$$

- We obtain the following volatility smile:

Obviously, the fair value of the variance swap is $\mathbb{E}\left[\langle X \rangle_T\right] = 0.04$

Heston formula for expected volatility

- By a well-known formula

$$\sqrt{y} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-\lambda y}}{\lambda^{3/2}} d\lambda$$

- Then, taking expectations

$$\mathbb{E}\left[\sqrt{\langle X \rangle_T}\right] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - \mathbb{E}\left[e^{-\lambda \langle X \rangle_T}\right]}{\lambda^{3/2}} d\lambda$$
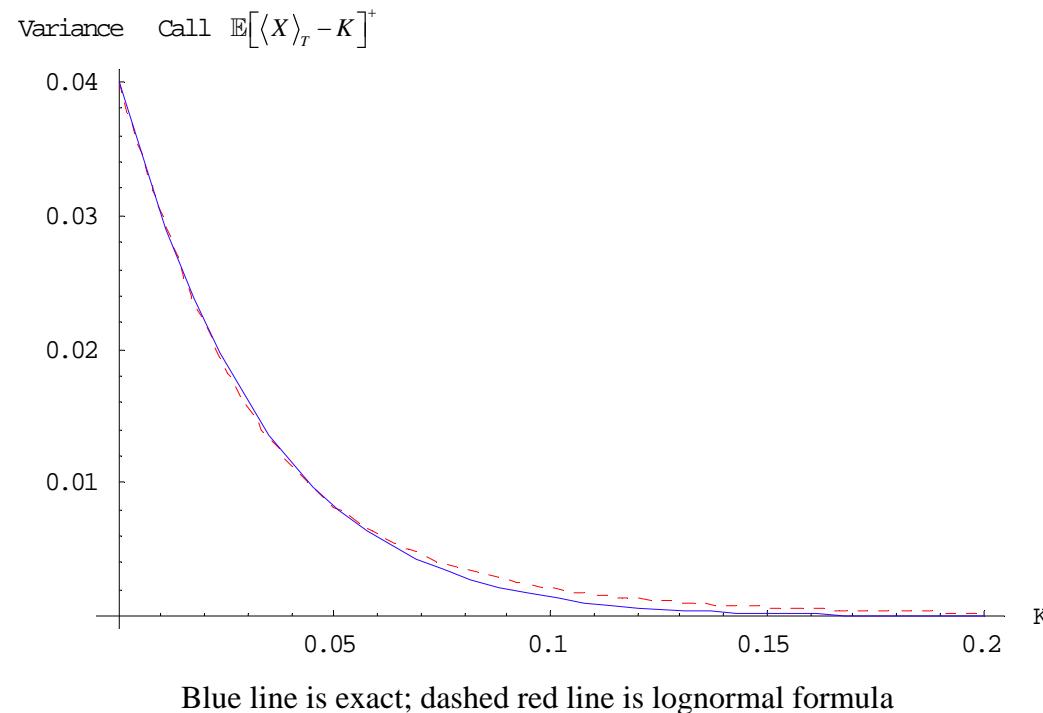
- We know the Laplace transform $\mathbb{E}\left[e^{-\lambda \langle X \rangle_T}\right]$ from the CIR bond formula.
- So we can also compute expected volatility explicitly in terms of the Heston parameters.
- With the BCC parameters, we obtain

$$\mathbb{E}\left[\sqrt{\langle X \rangle_T}\right] = 0.187429$$

- Our lognormal approximation then has parameters $s^2 = 0.129837;\ \mu = -1.73928$
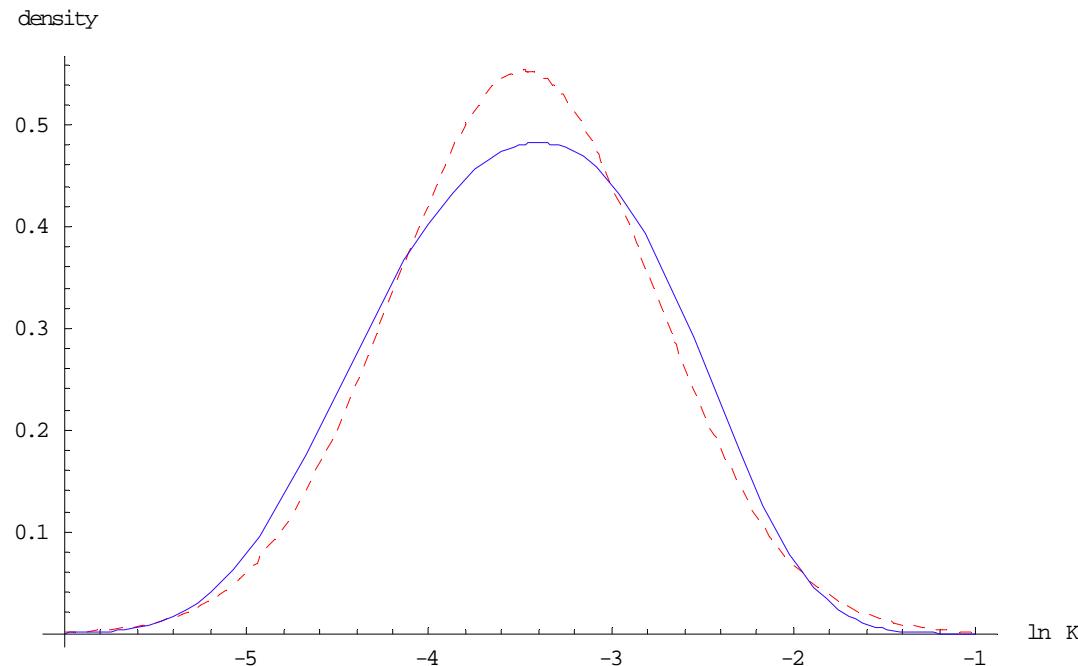
1 Year options on variance

- Now we value one year call options on variance (quadratic variation)
  - Exactly in the Heston model using BCC parameters
  - Using our simple Black-Scholes style formula with $s^2 = 0.129837$; $\mu = -1.73928$
- Results are as follows:

Variance Call $\mathbb{E}\left[\langle X \rangle_T - K\right]^+$

Blue line is exact; dashed red line is lognormal formula

pdf of quadratic variation

- Equivalently we can plot the pdf of the log of quadratic variation
  - Exactly in the Heston model using BCC parameters
  - Using our lognormal approximation
- Results are as follows:

Blue line is exact Heston pdf; dashed red line is lognormal approximation

Corollary

- We note that even when our assumptions on the volatility dynamics are quite different from lognormal (*i.e.* Heston), results are good for practical purposes.

- In practice, we believe that volatility dynamics are lognormal so results should be even better!

- So, if we know the convexity adjustment or equivalently, if we have market prices of variance and volatility swaps, we can use our simple lognormal model to price any (European-style) claim on quadratic variation with reasonable results.

Are volatility option prices uniquely determined by European option prices?

- We know from Carr and Lee (and then from Friz and Gatheral) that the prices of options on quadratic variation are uniquely determined if the correlation between volatility moves and moves in the underlying is zero.

- Moreover, we showed how to retrieve the pdf of quadratic variation from option prices under this assumption.

- What happens if correlation is not zero?

The Friz inversion algorithm

- We recall from Hull and White that in a zero-correlation world, we may write

$$c(k) = \int dy \, g(y) \, c_{BS}(k, y)$$

where $y := \sigma_{BS}^2 T$ is the total variance. In words, we can compute an option price by averaging over Black-Scholes option prices conditioned on the BS total implied variance.

- We assume that the law of $\log\left(\sqrt{\mathbb{E}\left[\langle X \rangle_T\right]}\right)$ is given by $\sum_i p_i \, \delta_{z_i}$

- Then

$$c(k) = \sum_i p_i \, c_{BS}(k, e^{2z_i})$$

- By construction, the mean and variance of the approximate lognormal pdf match the mean and variance of the true pdf. We thus use

$$q_i \propto \frac{1}{\sqrt{2\pi s^2}} e^{-(z_i - \mu)^2 / 2s^2}$$
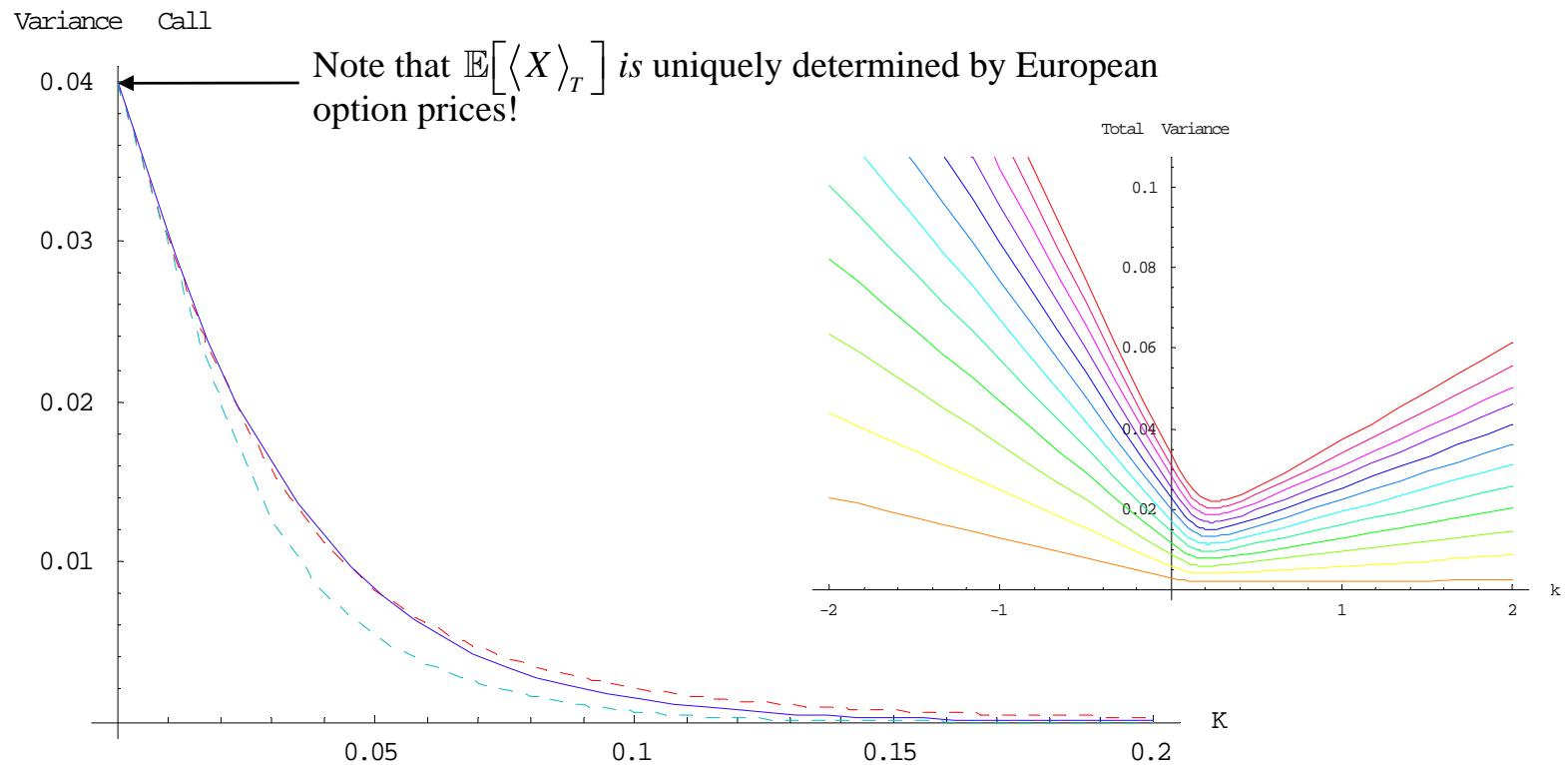
as an initial guess to the p_i and minimize the objective

$$\sum_j \left[\left\{\sum_i p_i \, c_{BS}(k_j, e^{2z_i})\right\} - c(k_j)\right]^2 + \beta \, d(p, q)$$

where $d(p,q)$ is some measure of distance such as relative entropy.

A local volatility computation

- We generate European option prices from the Heston model with BCC parameters and compute local volatilities.

- We use Monte Carlo simulation to compute the payoff of an option on quadratic variation on each path.

Note that $\mathbb{E}\left[\left\langle X \right\rangle_T\right]$ *is* uniquely determined by European option prices!
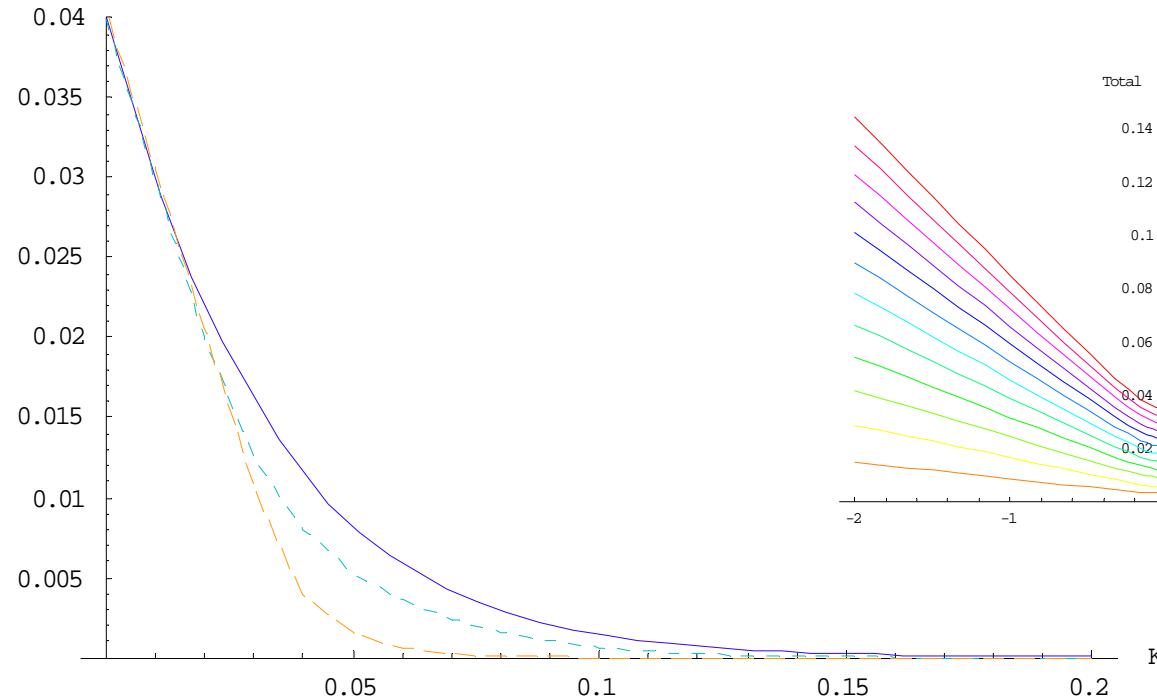
- Local volatility (the green curve) underprices volatility options.
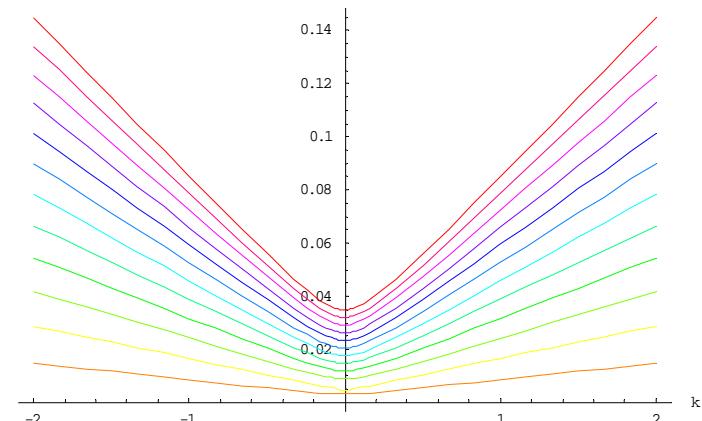
What happens if we change the correlation?

- Regenerate European option prices from the Heston model with BCC parameters but $\rho = 0$ and recompute local volatilities.

- Heston exact results and the lognormal approximation are both insensitive to the change in correlation. What about the local volatility approximation?

- The local volatility result (orange line) with $\rho = 0$ is lower still.

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A comment

- As Dupire has pointed out, the zero correlation assumption is very strong.
  - Local volatility is a diffusion process that is consistent with all given European option prices. But volatility moves and stock price moves are (locally) perfectly correlated.
- Given the prices of European options of all strikes and expirations, even restricting ourselves to diffusion processes, the pricing of volatility options is not unique.

- Dupire's recent construction of upper and lower bounds on the price of an option on volatility suggests the following conjecture:

A conjecture

- *Given the prices of European options of all strikes and expirations, of all possible underlying diffusions consistent with these option prices, the lowest possible value of a volatility option is achieved by assuming local volatility dynamics.*

Dupire's method for valuing VIX futures

- Roughly speaking, at time T_1 ,the VIX futures pays

$$\sqrt{\mathbb{E}_{T_1}\left[\langle X\rangle_{T_1,T_2}\right]} =: Y_1$$

where $\langle X\rangle_{T_1,T_2} := \langle X\rangle_{T_2} - \langle X\rangle_{T_1}$ is quadratic variation between T_1 and T_2 .
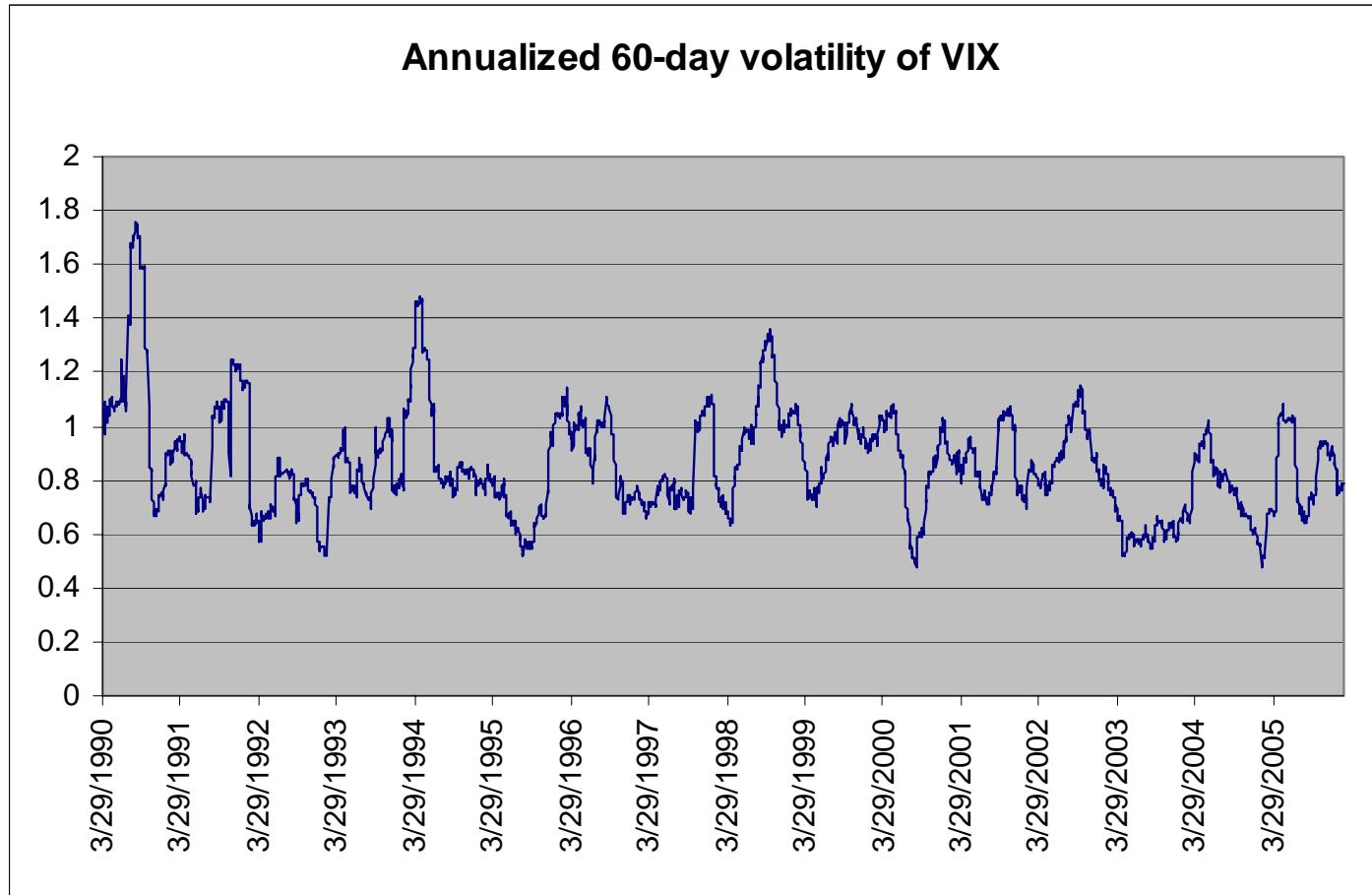
- Also, as before

$$\mathbb{E}_t\left[Y_1\right]^2 = \mathbb{E}_t\left[Y_1^2\right] - \text{var}\left[Y_1\right]$$

- We know $\mathbb{E}_t\left[Y_1^2\right] = \mathbb{E}_t\left[\langle X\rangle_{T_1,T_2}\right]$ in terms of the T_1 and T_2 log-strips.
- $\text{var}\left[Y_1\right]$ is estimated as the historical variance of VIX futures prices.

- The fair value of the VIX future is then given by $\mathbb{E}_t\left[Y_1\right] = \sqrt{\mathbb{E}_t\left[Y_1^2\right] - \text{var}\left[Y_1\right]}$

- A practical problem is that the VIX futures don't trade enough to give accurate historical vols.

Estimating volatility of volatility

- We can compute historical volatility of the VIX

Annualized 60-day volatility of VIX

- …pretty stable at around 80-100%

Then we apply the volatility envelope

- We note that if $dv = -\lambda(v - \bar{v})\,dt + noise$, variance swaps should behave as

$$W_T := \mathbb{E}\left[\langle X \rangle_T\right] \sim (v - \bar{v})\frac{1 - e^{-\lambda T}}{\lambda} + \bar{v}\,T$$

- Then assuming that changes in instantaneous volatility drive most of the changes in the volatility surface, we obtain

$$\Delta W_T \sim \Delta v\,\frac{1 - e^{-\lambda T}}{\lambda}$$

and changes in forward starting variance swaps are given by

$$\Delta W_{T_1,T_2} \sim \Delta v\,\frac{e^{-\lambda T_1} - e^{-\lambda T_2}}{\lambda} \simeq e^{-\lambda T_1}\Delta v\,(T_2 - T_1)$$

- We see that the volatility of Y_1^2 should decay exponentially with maturity T_1. We can identify changes in v with changes in the VIX.
- We can estimate λ from the term structure of skew for example.
- Or we can use an empirically estimated volatility envelope to decay volatility as a function of time to maturity.

VIX Futures Fair Value

Ticker	Expiration	Days	Risk Free	Upper Bound	Volatility of Vix		Fair Value	Futures Price	Fair – Futures
UXV5	10/19/05	7	3.32%	153.59	56.54	2	153.12	145.00	8.12
	11/19/05*		3.40%		Historical				
UXX5	11/16/05	35	3.40%	158.18	53.06	2	156.06	151.00	5.06
	12/17/05*		3.42%		Historical				
UXG6	2/15/06	126	3.76%	162.44	46.50	2	156.50	159.50	-3.00
	3/18/06*		3.86%		Historical				
UXK6	5/17/06	217	0.28%	172.82	39.67	2	164.93	166.00	-1.07
	6/17/06*		0.52%		Historical				

Decaying volatilities

* Options of these additional maturities are used to compute the Fair Price of the corresponding futures contracts.

VXB Fast	VXB Spot	Fast – Spot
153.91	154.00	-0.09

Australia 61 2 9777 8600   Brazil 5511 3048 4500   Europe 44 20 7330 7500   Germany 49 69 920410
Hong Kong 852 2977 6000 Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2005 Bloomberg L.P.
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<HELP> for explanation.   P181 Index **FVD**

VIX Futures Fair Value

Ticker	Expiration	Days	Risk Free	Upper Bound	Volatility of Vix		Fair Value	Futures Price	Fair − Futures
UXV5	10/19/05	7	3.32%	153.59	245.72	1	145.00	145.00	0.00
	11/19/05*		3.40%		Implied				
UXX5	11/16/05	35	3.40%	158.18	98.47	1	151.00	151.00	0.00
	12/17/05*		3.42%		Implied				
UXG6	2/15/06	126	3.76%	162.44	32.56	1	159.50	159.50	0.00
	3/18/06*		3.86%		Implied				
UXK6	5/17/06	217	0.28%	172.82	36.82	1	166.00	166.00	0.00
	6/17/06*		0.52%		Implied				

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Remarks

- Note that historical and implied volatilities on Bloomberg are consistent with the historical volatility of the VIX.

- So we have a practical way of estimating volatility of volatility.

- And we can compute variance swaps from the log-strips.

- We can then use our simple lognormal model to price any European-style claim on quadratic variation.

Trading volatility of volatility: implied vol of vol

- We have shown how volatility of volatility might be estimated from historical data or alternatively, from a knowledge of the variance-volatility convexity adjustment.

- In practice, even where the convexity adjustment

$$\sqrt{\mathbb{E}\left[\langle X \rangle_T\right]} - \mathbb{E}\left[\sqrt{\langle X \rangle_T}\right]$$

  is traded, there is a bid-offer spread.

- So volatility of volatility becomes another implied parameter that is effectively quoted and traded with its own bid-offer spread.

- An example: the market for the one-year SPX variance-volatility convexity adjustment is currently around 0.80-1.30 with the variance swap at 15.7% (mid-market).

  - Under our lognormal assumption,

$$s^2 = 2\log\left(\frac{\sqrt{\mathbb{E}\left[\langle X \rangle_T\right]}}{\mathbb{E}\left[\sqrt{\langle X \rangle_T}\right]}\right)$$

  - Then the volatility of volatility s is roughly 0.32 bid at 0.42 offered.
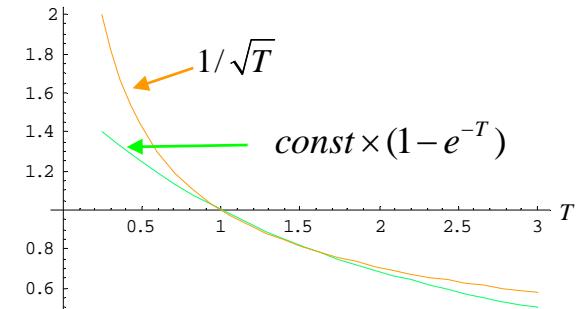
# Comparing implied vol of vol with historical

- 0.32 bid @ 0.42 offered is the effective market for (average) vol of vol of one-year SPX implied volatility.

- According to our earlier analysis, the volatility of one-year volatility should be given by

$$\Delta \sigma_1^{\,2} \sim \Delta VIX \frac{1 - e^{-\lambda}}{\lambda}$$

- From the Bloomberg historical VIX futures volatilities, we estimate $\lambda \sim 1$

- Note that the shape of $(1 - e^{-T})$ is not very different from $1/\sqrt{T}$

- Then we have

$$\Delta VIX \sim \frac{\lambda}{1 - e^{-\lambda}} \Delta \sigma_1 \simeq 1.6 \, \Delta \sigma_1$$

- This translates to an implied spot VIX volatility bid-offer spread of 0.51-0.67.

- ... not inconsistent with historical VIX implied volatility!

Summary

- We showed how to compute expected quadratic variation for compound Poisson processes.

- We computed the magnitude of the jump correction to the diffusion-based valuation of variance swaps and suggested that at least for indices, jumps have little impact on the valuation.

- We described a simple lognormal model for estimating the value of options on volatility and related the parameters to the values of variance and volatility swaps.

- Under the zero correlation assumption, we showed how to recover the unique density of quadratic variation from European option prices.

- We investigated whether or not volatility option prices are in general uniquely determined by European option prices and showed that they are not.

- We described Dupire's method for valuing VIX futures.

- Finally, we suggested how to estimate volatility of volatility.

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